Aerospace Research Center

GUIDANCE APPLICATIONS OF A QUASI-OPTIMUM CONTROL TECHNIQUE

Contract No. NAS 8-20311

FINAL REPORT

Contributors:

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Submitted to:
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
GEORGE C. MARSHALL SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA 35812

November 15, 1967

AEROSPACE RESEARCH CENTER
GENERAL PRECISION SYSTEMS INC
KEARFOTT GROUP
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ABSTRACT

This report describes the application of two versions of a quasi-optimum control technique, developed at the General Precision Aerospace Research Center, to two guidance problems.

The first application is that of developing an explicit guidance technique for orbital injection, and comparing this technique to other guidance schemes, particularly the Iterative Guidance Mode (IGM) developed at the NASA Marshall Space Flight Center. It was found that General Precision's quasi-optimum control technique results in orbital injection at an earlier time than obtained by any of the other techniques to which it was compared, and it gives excellent results even when the IGM technique fails. The particular merit of the General Precision technique, however, is significant only in flights which are of longer duration than typical of near-term Saturn V flights.

The second application is that of developing a guidance technique for very low-thrust vehicles. As a result of normal computational errors (truncation and round-off) the results achieved are inconclusive, but a slight advantage is seen by use of the quasi-optimum control technique.

CONTENTS

1. INTRODUCTION AND SUMMARY

The general problem of optimum guidance is the determination of a steering law by which a vehicle, starting in an arbitrary state is brought to a desired terminal manifold in an optimum fashion. The optimum is usually defined as requiring a minimum fuel expenditure, thereby maximizing the payload which can be carried.

The determination of the optimum steering is a well-known problem in the calculus of variations and can be reduced to the solution of a two-point boundary-value problem of ordinary (nonlinear) differential equations. Effective computational techniques for solving such two-point boundary value problems are available but require the use of a large, high-speed digital computer and are not suitable for real-time on-board implementation (explicit guidance).

The need for approximate techniques which are capable of being implemented on board the vehicle but which nevertheless yield nearly optimum performance has been recognized for a number of years and is currently receiving attention by several investigators.

Several versions of a quasi-optimum control technique [1, 2] which has given evidence of satisfying these requirements have been developed at the Research Center of the General Precision Systems Inc., Kearfott Group, and various applications of the technique [3, 4] have been reported in the technical literature. The objective of this investigation was to determine whether the technique is useful in two applications of interest to the Marshall Space Flight Center.

The application studied more recently was the use of one version of the General Precision quasi-optimum control technique to improve the performance of the MSFC - developed iterative guidance mode (IGM) method of explicit guidance for orbital injection, and to compare the resulting performance with other explicit guidance techniques. These included a flat earth model with a constant gravitational field over the remaining time-to-go (simplified process for the quasi-optimum guidance law) and a polynomial gravity model currently under study at MSFC. The results of the investigation demonstrated that the General Precision quasi-optimum control technique investigated is always better than the other guidance techniques to which it was compared. The amount of improvement over the

IGM is not significant in relatively flat, short duration (200 sec.) trajectories which are typical of normal Saturn flights, but becomes more evident in steep trajectories, in which the General Precision technique (as well as others) are significantly better than the IGM technique and can achieve as much as a 10 percent reduction in fuel consuption. The superiority of the General Precision technique to the others is even more evident in trajectories of longer duration (which are not typical of current Saturn missions, but may well represent missions of the next decade). In relatively flat trajectories of 500 second duration, for example, the General Precision quasi-optimum technique provided as much as an eight percent reduction in fuel consumption compared to the IGM.

The most remarkable performance of the General Precision quasi-optimum technique was achieved on a trajectory of 1000 seconds optimum duration. The General Precision quasi-optimum technique achieved the desired orbit in 1001.56 seconds which the IGM and polynomial gravity steering laws failed to achieve the desired orbit and the simplified steering law inserted the vehicle into the desired orbit in 1,231 seconds.

An earlier study was made of the application of a different version of the quasi-optimum control technique on the problem of orbital transfer in a vehicle using extremely low thrust. A typical mission was visualized as that of (1) ascending from a near circular orbit around the earth to (2) a heliocentric orbit around the Sun, and finally (3) descent to a near circular orbit around a prescribed planet. The objective of the guidance technique was to accomplish the mission with minimum fuel expenditure. In this investigation we had considered only the first part of the problem, namely the development of the steering law which optimizes the trajectory from an arbitrary starting state or the beginning of each phase to an arbitrary state at the end of that phase.

Owing to difficulties which we believe are attributable to normal digital computer round-off and truncation errors, the results obtained are inconclusive. A reprogramming of the algorithm to avoid unnecessary computations and the attendant computational errors thereof should be undertaken prior to the resumption of this study.

2. THEORY

The theoretical basis of the quasi-optimum control techniques studied in this investigation was established at the General Precision Aerospace Research Center and studied under Contracts NAS 2-2648 [2] and NAS 2-3636 with the NASA Ames Research Center. Details of the theory and a number of examples of application are contained in References [1-5]. For completeness, however, a brief review of the theory is included here.

Consider the problem of minimization of $x_0(T)$ for the system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x = \{x_0, \ldots x_n\} \tag{2-1}$$

where $x(0)$ is known, the boundary conditions are of the form

$$\varphi_i(x(T)) = 0 \qquad i = 1, 2, \ldots, m \le n \tag{2-2}$$

T is free and $u(t)$ must be a member of a given set Ω.

It is well known that if an optimal u^* exists then u^* maximizes the Hamiltonian

$$H(x, u^*, p) = \max_{u \in \Omega} H(x, u, p) = \max_{u \in \Omega} p' f(x, u) \tag{2-3}$$

where $H(x, u^*, p) = 0$. Consequently, u^* can be obtained as a function of the adjoint state $p = \{p_0, \ldots, p_n\}$ and the process state x:

$$u^* = \sigma(p, x) \tag{2-4}$$

The adjoint vector $p(t)$ is governed by

$$\dot{p}(t) = -H_x \tag{2-5}$$

with

$$p(T) = \left[\begin{array}{c} 1 \\ \hline \Phi' \lambda \end{array}\right] \begin{array}{l} 1 \\ n \end{array} \tag{2-6}$$

where H_x is the gradient of the Hamiltonian with respect to x, λ is a vector of m constants and Φ is defined by

$$\Phi = \left[\frac{\partial \varphi_i}{\partial x_j}\right] \qquad i = 1, 2, \ldots, m \; ; \quad j = 0, 1, \ldots, n$$

Let $X(t)$, $P(t)$ be solutions of the two-point boundary-value problem (2-1) - (2-6) and let

$$x(t) = X(t) + \xi(t) \tag{2-7}$$

be the state in an altered problem (e.g., with different initial or terminal conditions or different dynamics), and suppose that ξ is a small quantity. As a result of the change ξ in x, the adjoint vector will change by an amount ψ, i.e.,

$$p(t) = P(t) + \psi(t) \tag{2-8}$$

Substitution of (2-7) and (2-8) into (2-1) and (2-5) results in

$$\begin{aligned} \dot{X} + \dot{\xi} &= H_p = H_P + H_{XP}\xi + H_{PP}\psi + O(\xi^2) \\ \dot{P} + \dot{\psi} &= -H_x = -H_X - H_{XX}\xi - H_{PX}\psi + O(\xi^2) \end{aligned} \tag{2-9}$$

where H_P and H_X are the gradients of the Hamiltonian evaluated at $p = P$, $x = X$ and

$$H_{XP} = \left[\frac{\partial^2 H}{\partial x_j \partial p_i}\right]_{\substack{x=X \\ p=P}} \qquad H_{PX} = \left[\frac{\partial^2 H}{\partial p_j \partial x_i}\right]_{\substack{x=X \\ p=P}} = H'_{XP}$$

$$H_{PP} = \left[\frac{\partial^2 H}{\partial p_j \partial p_i}\right]_{\substack{x=X \\ p=P}} \qquad H_{XX} = \left[\frac{\partial^2 H}{\partial x_j \partial x_i}\right]_{\substack{x=X \\ p=P}}$$

Since X and P are solutions to (2-1) - (2-6) it follows that

$$\dot{X} = H_P \qquad \dot{P} = -H_X$$

and hence, after dropping terms of $O(\xi^2)$, (2-9) becomes

$$\begin{aligned} \dot{\xi} &= H_{XP}\xi + H_{PP}\psi \\ \dot{\psi} &= -H_{XX}\xi - H_{PX}\psi \end{aligned} \tag{2-10}$$

These differential equations are linear, and can be integrated to give

$$\begin{aligned} \xi(T) &= \Phi_{11}(T,t)\xi(t) + \Phi_{12}(T,t)\psi(t) \\ \psi(T) &= \Phi_{21}(T,t)\xi(t) + \Phi_{22}(T,t)\psi(t) \end{aligned} \tag{2-11}$$

where $\Phi_{ij}(T,t)(i,j = 1,2)$ are the $(n+1) \times (n+1)$ blocks of the $(2n+2) \times (2n+2)$ fundamental (transition) matrix of (2-10). Our objective is to find a relationship between the correction $\psi(t)$ to the adjoint vector $p(t)$ and the deviations $\xi(t)$ of the state from $X(t)$. To do this it is necessary to eliminate $\xi(T)$ and $\psi(T)$ from (2-11) by use of the boundary conditions for the original (exact) problem (2-1) - (2-6).

Consider a state variable x_i fixed at $t = T$. Then

$$\begin{aligned} x_i(T + dT) &= x_i(T) + \dot{x}_i(T)dT \\ &= X_i(T) + \xi_i(T) + \dot{X}_i(T)dT + \dot{\xi}_i(T)dT \end{aligned} \tag{2-12}$$

The last term is a second-order infinitesimal and can be dropped. If in the simplified problem the constraint is satisfied by X_i at time T, then in the exact problem the constraint must be satisfied at $T + dT$. Thus we must have $x_i(T + dT) = c_i = X_i(T)$ and hence (2-12) becomes:

$$\xi_i(T) = -\dot{X}_i\, dT \quad \text{for} \quad X_i(T) \quad \text{fixed} \tag{2-13}$$

Finally, we must have

$$dH = \xi' \frac{\partial H}{\partial X} + \psi' \frac{\partial H}{\partial P}$$

$$= -\dot{P}'\xi + \dot{X}'\psi = 0 \tag{2-15}$$

Equations (2-12) - (2-15) give $n+2$ relations. Since dT is an additional variable, there are just enough equations needed to solve (2-11) for $\psi(t)$ as a function of $\xi(t)$. It is readily established that upon elimination of $\psi(T)$ and $\xi(T)$, a linear relation between $\psi(t)$ and $\xi(t)$ is obtained:

$$\psi(t) = M(t)\xi(t) \tag{2-16}$$

Upon differentiation of (2-16) and substitution of the result into (2-10), there results

$$(dM/dt + MH_{XP} + H_{XP}M + MH_{PP}M + H_{XX})\xi = 0$$

If this realtionship is to hold for any ξ, the matrix M must satisfy the matrix Riccati equation:

$$-dM/dt = MH_{XP} + H_{PX}M + MH_{PP}M + H_{XX} \tag{2-17}$$

It is evident that if M is a solution to (2-17) then M' is a solution to (2-17); thus the solution to (2-17) can be a symmetric matrix. In fact, when the adjoint vector p can be interpreted as the negative of the gradient of the optimum value of $x_0(T) = V$ [2], then

$$M = -\left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]_{x=X} \tag{2-18}$$

Because $\xi(t)$ is a change in $x(t)$ and $\psi(t)$ is the corresponding change in $p(t)$, it follows that

$$M = \left[\frac{\partial p_i}{\partial x_j}\right]_{x=X} \tag{2-19}$$

even when p cannot be interpreted as the negative gradient of V.

Constraints on the magnitude of the control variables, $|u_l| \le U_l$ for example, will generally lead to a control law which is a discontinuous function of the adjoint variables, ahd hence, in a strict sense not all the partial derivatives required in (2-10) or (2-17) will exist. The discontinuous control variables can be treated by introduction of impulses (delta functions) which arise upon differentiation.

In many problems it is possible to approximate the dynamic behavior of the process by a system of differential equations of considerably simpler form than those actually governing the process. Under favorable circumstances an analytic solution to the simplified problem can be found, but the use of the control law derived for the simplified process may not be entirely adequate for the exact dynamic model. If the neglected terms in the original dynamic model were accounted for approximately, however, it might be possible to improve performance to an acceptable level. The suboptimum control equations of the previous section provides a method of so doing.

The control u is generated as a function of the state x and the adjoint state p. Instead of using the exact (unknown) relation $p = p(x)$ between the adjoint state and the process state the adjoint state is approximated by

$$p = P + M\xi \tag{2-20}$$

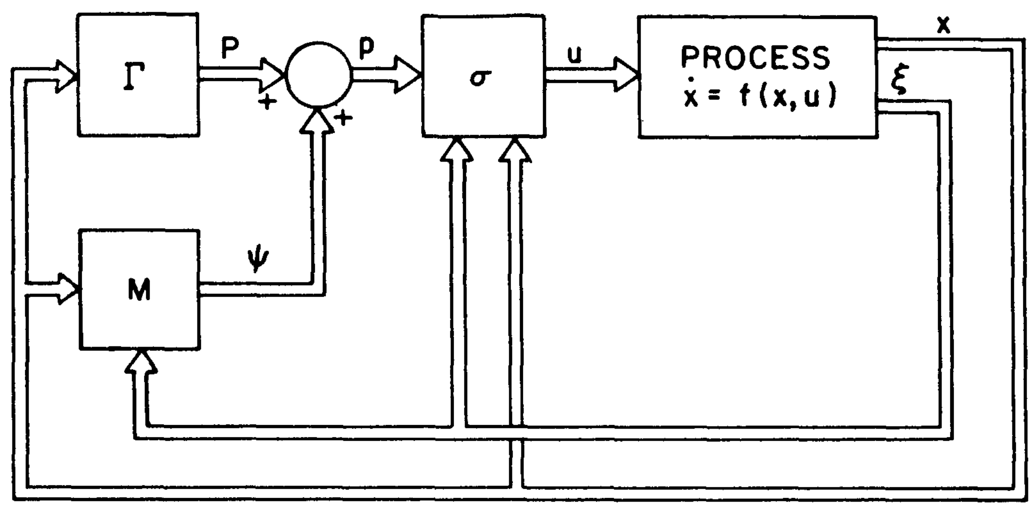
where $P = P(X)$ is obtained as the analytic solution of the simplified problem and

$$M = M(X)$$

is the matrix M of the previous section, expressed as a function of the state X instead of time. (To eliminate time from M, it is necessary to express time in terms of the state variables along the optimum trajectory of the simplified system.)

The control system, using the optimum transformation from p to u, but the approximate relation (2-20) for the transformation from x to p has the configuration of Figure 2.1 .

In many cases it will not be possible to obtain an explicit analytical expression for $M(X)$. Numerical integration of the Riccati equation, using a function of the state rather than time as the independent variable, and analytical approximation of the result may prove feasible. An alternative procedure is to use an asymptotic solution of (2-17), obtained by setting dM/dt to zero and solving the resulting algebraic system. This was the approach used in the present study.



STRUCTURE OF
QUASI-OPTIMUM CONTROL SYSTEM
FIGURE 2.1

A second method of employing the suboptimum control technique is based on the interpretation of (2-19) as the Jacobian matrix of p with respect to x. As a consequence of this interpretation it follows that

$$\dot{p} = \left[\frac{\partial p_i}{\partial x_j} \right] \dot{x} = M\dot{x} \tag{2-21}$$

provided that the partial derivatives in the matrix M are evaluated at the true state x of the process. Thus the adjoint vector can be obtained by integration of (2-21);

$$p = p_0 + \int_{t_0}^{t} M\dot{x}\, dt \tag{2-22}$$

This relation leads to a control system with the configuration shown in Figure 2.2(a). It is noted that the derivatives of the state variables instead of the state variables themselves are the quantities fed back. Hence this technique is particularly applicable to problems in inertial guidance, where the principal sensors are accelerometers.

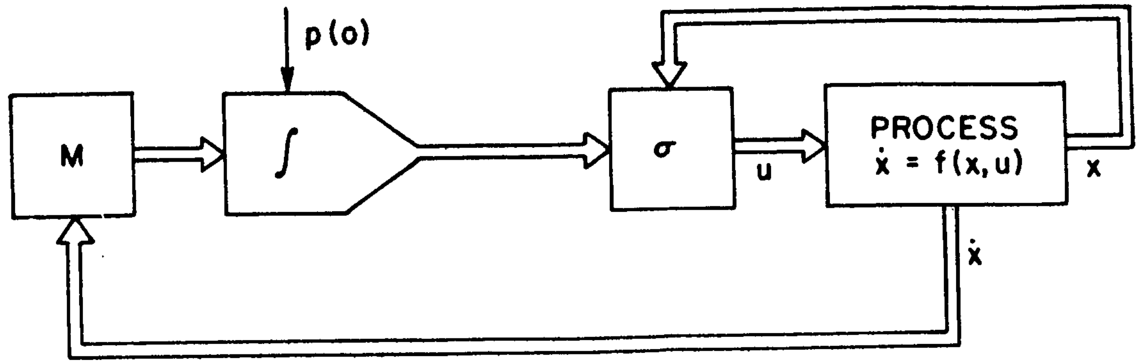
In the event that $\dot{x}$ cannot be sensed, an alternative configuration can be obtained by partial integration of (2-22) :

$$p = p_0 + Mx - M_0 x_0 - \int_{t_0}^{t} \dot{M} x\, dt \tag{2-23}$$

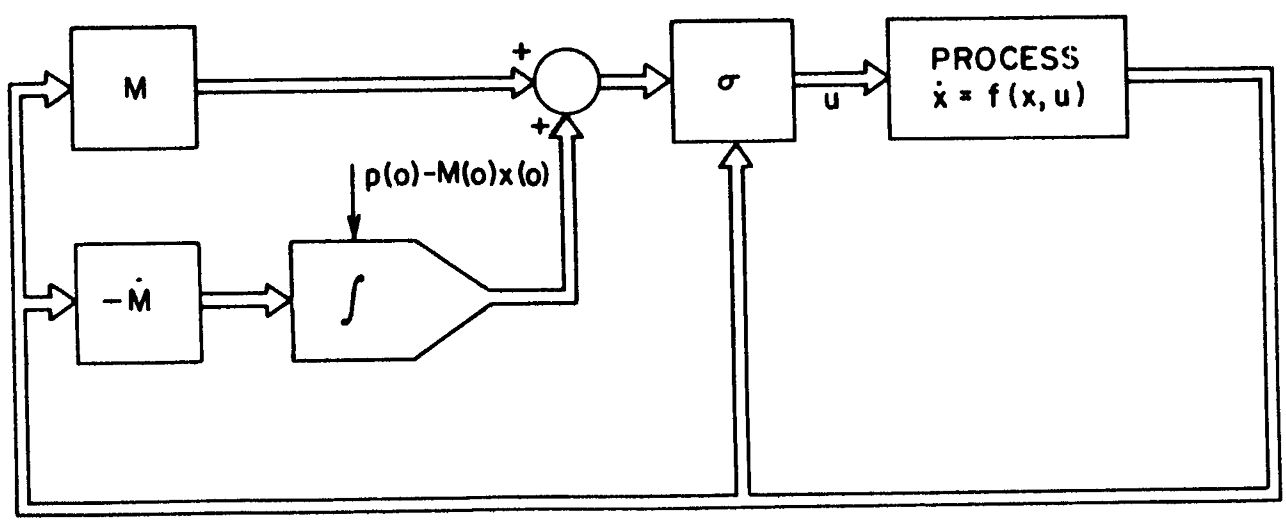
The right-hand side of (2-17) is used for $-\dot{M}$ in (2-23). The control system configuration corresponding to (2-23) is shown in Figure 2.2(b); it is seen that only the state x is required in the controller.

In either implementation the matrix M would be generated by real-time integration of (2-17) with the nominal initial condition M_0 , and the nominal initial adjoint state P_0 would be used. Thus to achieve near-optimum performance, the actual initial state x_0 should be reasonably close to the nominal initial state X_0 for which M_0 and P_0 were computed. If the closed-loop system is asymptotically stable, however, the effects of using initially incorrect values of M_0 and p_0 will be only transient.

The first method described was used in the application to IGM improvement and the second method was used in the low-thrust guidance application.



(a) ẋ MEASURABLE



(b) ẋ NOT MEASURABLE

SUBOPTIMUM CONTROL BASED ON p(0) , M(0)
FIGURE 2.2

3. APPLICATION TO IGM IMPROVEMENT

3.1 Exact Problem Formulation

This application is concerned with the improvement of the MSFC Iterative Guidance Mode [Ref. 6,7] (IGM) computation by use of Friedland's quasi-optimum control technique in which the optimum control law for a simplified model of the process is linearly corrected to account for the difference between the simplified model and the actual process. In this application the simplified model is obtained by assuming that the gravitational acceleration is constant in magnitude and direction over the remaining time-to-go.

The guidance problem to which the quasi-optimum technique is to be applied consists of placing a payload in a fixed orbit with minimum fuel expenditure using a booster which has a constant thrust magnitude. The assumptions made in the formulation of the exact problem are:

(a) The gravitational acceleration is due to a spherical earth; oblateness is not considered.

(b) Vehicle motion occurs in a vacuum; no atmospheric drag is included.

(c) Vehicle thrust and mass-flow rate are constants.

(d) Motion of the vehicle is confined to a single plane.

The resultant system is governed by the following differential equations in an earth centered cartesian coordinate system.

$$\dot{x}_0 = - \dot{m}$$

$$\dot{x}_1 = \frac{F}{x_0} u_1 - g_x^* + x_5$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_3 = \frac{F}{x_0} - g_y^* + x_6$$

$$\dot{x}_4 = x_3$$

$$\dot{x}_5 = \mu \frac{\left\{2x_1x_2^2 - x_1x_4^2 + 3x_2x_3x_4\right\}}{\left\{x_2^2 + x_4^2\right\}^{5/2}}$$

$$\dot{x}_6 = \mu \frac{2x_3x_4 - x_2^2x_3 + 3x_1x_2x_4}{\left\{x_2^2 + x_4^2\right\}^{5/2}} \qquad (3\text{-}1)$$

where

x_0 = vehicle mass

g_x^*, g_y^* are the x and y components of gravitation acceleration which are constant in magnitude and direction during the remaining time-to-go.

u_1, u_2 are the direction cosines of thrust acceleration

F = vehicle thrust

μ = gravitation constant

x_5, x_6 are terms used to account for the difference between the approximate (constant) gravitational field and the true field.

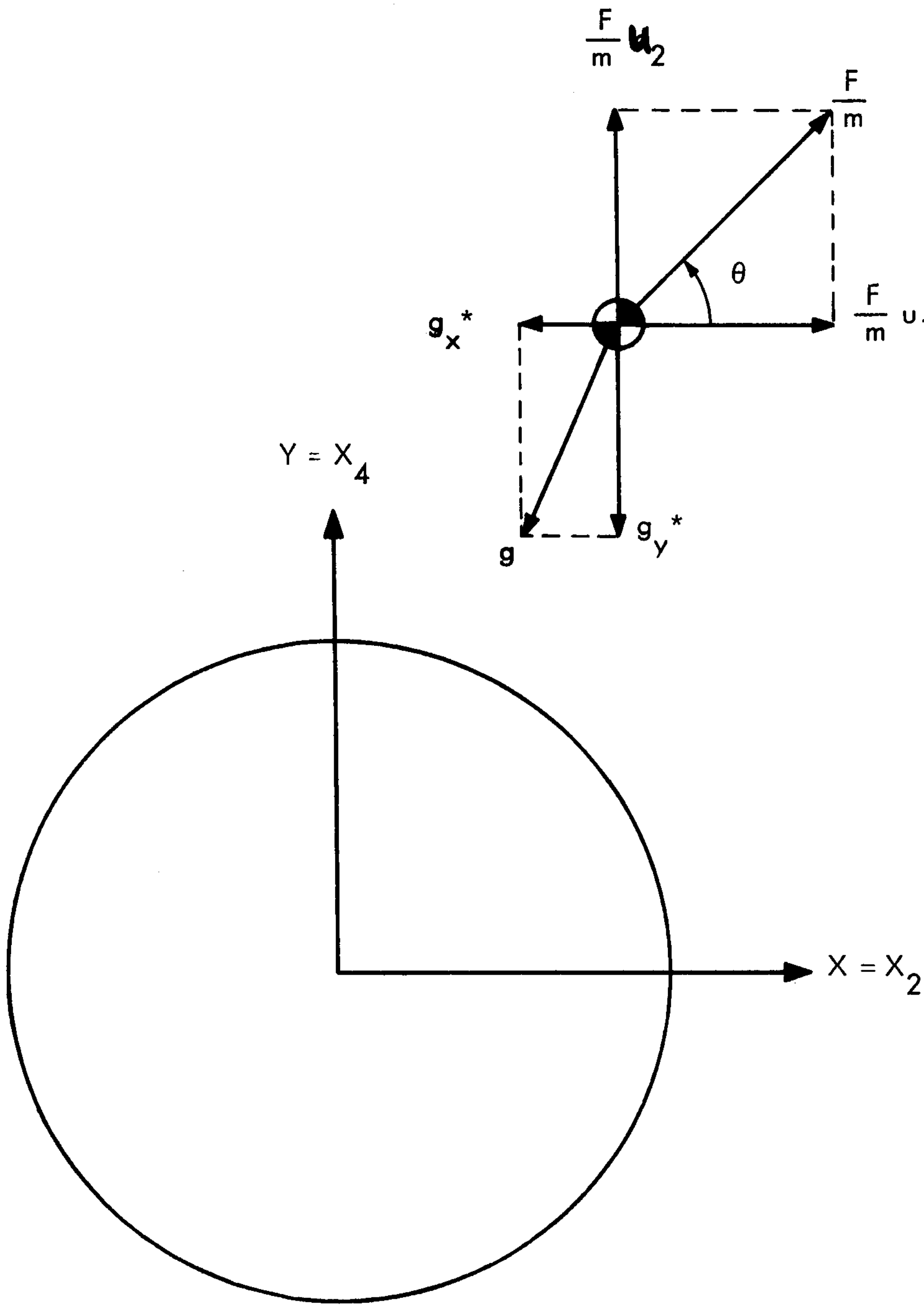
The physical situation is illustrated in Figure 3-1.

The problem then is to determine the control vector $u = \{u_1, u_2\}$ subject to the constraint

$$u_1^2 + u_2^2 = 1 \qquad (3\text{-}2)$$

that takes the process from some current state[1] $x(t)$ to a final state $x(T)$ such that the performance index $x_0(T)$ is a minimum and the remaining states satisfy the boundary conditions

1 The current time is denoted by the variable t, terminal time by T; time when it is used as an independent variable is denoted by τ, i.e., $t < \tau < T$.



VEHICLE MOTION IN UNIFORM GRAVITATIONAL FIELD
FIGURE 3-1

$$
\begin{aligned}
x_1^2(T) + x_3^2(T) &= V^2 \\
x_1(T)x_2(T) + x_3(T)x_4(T) &= R \cdot V \\
x_2^2(T) + x_4^2(T) &= R^2 \\
\{x_5(T) - g_x^* \}^2 + \{x_6(T) - g_y^*\}^2 &= g_R
\end{aligned}
\tag{3-3}
$$

where

V = orbital velocity

R = orbit radius

g_R = magnitude of gravitational acceleration at orbital altitude

$R \cdot V$ = position - veloctiy dot product $\{R \cdot V = 0$ for circular orbits$\}$

The terminal time T is free.

The Hamiltonian for this problem is

$$
h = -p_0\dot{m} + p_1\left(\frac{F}{x_0}u_1 - g_x^* + x_5\right) + p_2x_1 + p_3\left(\frac{F}{x_0}u_2 - g_y^* + x_6\right) + p_4x_3 + p_5\dot{x}_5 + p_6\dot{x}_6 \tag{3-4}
$$

Maximization of the Hamiltonian with respect to u_1 and u_2 subject to (3-2) results in the following steering law

$$
\begin{aligned}
u_1 &= \frac{p_1}{\left(p_1^2 + p_3^2\right)^{1/2}} \\
u_2 &= \frac{p_3}{\left(p_1^2 + p_3^2\right)^{1/2}}
\end{aligned}
\tag{3-5}
$$

Using these values of u_1 and u_2 yields

$$
h = -p_0\dot{m} + \frac{F}{x_0}\left(p_1^2 + p_3^2\right)^{1/2} + p_1(-g_x^* + x_5) + p_2x_1 + p_3(-g_y^* + x_6) + p_4x_3 + p_5\dot{x}_5 + p_6\dot{x}_6 \tag{3-6}
$$

3.2 Simplified Problem

In this application of the quasi-optimum technique the simplified problem is obtained by assuming x_5 and x_6 are zero. It is observed that the exact and simplified problem differ only by the presence of x_5 and x_6 in the former. These terms must be included in the exact problem to account for the difference between the approximate (constant) gravitational field and the true field. Since the assumption of uniform gravitation is known to be reasonably accurate we conclude that x_5 and x_6 are small quantities and it is reasonable to use the solution of the simplified problem as the basis for an approximate solution to the exact problem. Thus we select as the state of the simplified process

$$X = \{x_0, x_1, x_2, x_3, x_4, 0, 0\} \tag{3-7}$$

Then

$$\xi = \{\xi_0, 0, 0, 0, 0, x_5, x_6\}$$

(Note that ξ_0 is the change in the performance index due to the simplification.)

The simplified process is governed by the following differential equations

$$\begin{aligned}
\dot{X}_0 &= -\dot{m} \\
\dot{X}_1 &= \frac{F}{X_0}u_1 - g_x^* \\
\dot{X}_2 &= X_1 \\
\dot{X}_3 &= \frac{F}{X_0}u_2 - g_y^* \\
\dot{X}_4 &= X_3
\end{aligned} \tag{3-8}$$

It is desired to minimize $X_0(T)$ subject to

$$\begin{aligned}
&X_1^2(T) + X_3^2(T) - V^2 = 0 \\
&X_2^2(T) + X_4^2(T) - R^2 = 0 \\
&X_1(T)X_2(T) + X_3(T)X_4(T) - V \cdot R = 0 \\
&P_1(T)X_3(T) + P_2(T)X_4(T) - P_3(T)X_1(T) - P_4(T)X_2(T) = 0
\end{aligned} \tag{3-9}$$

where $\{u_1, u_2\} \in \{\Omega \mid u_1^2 + u_2^2 = 1\}$

The Hamiltonian for the simplified problem is

$$H = -P_0\dot{m} + P_1\left[\frac{F}{X_0}u_1 - g_x^*\right] + P_2X_1 + P_3\left[\frac{F}{X_0}u_2 - g_y^*\right] + P_4X_2 \tag{3-10}$$

Maximization of the Hamiltonian with respect to u_1 and u_2 gives

$$\text{Max } H \Rightarrow \quad u_1 = \frac{P_1}{(P_1^2 + P_3^2)^{1/2}} \qquad u_2 = \frac{P_3}{(P_1^2 + P_3^2)^{1/2}} \tag{3-11}$$

Substituting u_1 and u_2 into the Hamiltonian gives

$$H = P_0\dot{m} + \frac{F}{X_0^2}\left[P_1^2 + P_3^2\right]^{1/2} - P_1g_x^* + P_2X_1 - P_3g_y^* + P_4X_3 \tag{3-12}$$

The corresponding adjoint equations are

$$\begin{aligned}
\dot{P}_0 &= \frac{F}{X_0^2}\left[P_1^2 + P_3^2\right]^{1/2} \\
\dot{P}_1 &= -P_2 \\
\dot{P}_2 &= 0 \\
\dot{P}_3 &= -P_4 \\
\dot{P}_5 &= 0
\end{aligned} \tag{3-13}$$

Let t denote the present time and let $t \leq \tau \leq T$. Then the solution of (3-13) is given by

$$\begin{aligned}
P_1(\tau) &= P_1(t) - (\tau - t)P_2(t) \\
P_2(\tau) &= P_2(t) \quad \text{constant} \\
P_3(\tau) &= P_3(t) - (\tau - t)P_4(t) \\
P_4(\tau) &= P_4(t) = \text{constant}
\end{aligned} \tag{3-14}$$

The optimum control law for the simplified process (3-8) becomes

$$u_1(\tau) = \cos\theta = \frac{P_1(t) - (\tau - t)P_2(t)}{\{[P_1(t) - (\tau - t)P_2(t)]^2 + [P_3(t) - (\tau - t)P_4(t)]^2\}^{1/2}}$$

$$u_2(\tau) = \sin\theta = \frac{P_3(t) - (\tau - t)P_4(t)}{\{[P_1(t) - (\tau - t)P_2(t)]^2 + [P_3(t) - (\tau - t)P_4(t)]^2\}^{1/2}} \tag{3-15}$$

Substituting (3-15) into (3-8) and integrating to the terminal time T, results in expressions for $X_1(T)$, $X_2(T)$, $X_3(T)$, $X_4(T)$, as functions of $T - t$ (time-to-go) and the initial conditions. The integrals are

$$X_1(T) - X_1(t) = F\int_t^T \frac{P_1(t) - (\tau - t)P_2(t)}{\{X_0(t) - (\tau - t)\dot{m}\}\{[P_1(t) - (\tau - t)P_2(t)]^2 + [P_3(t) - (\tau - t)P_4(t)]^2\}^{1/2}}\, d\tau - g_x^* \int_t^T d\tau$$

$$X_2(T) - X_2(t) = \int_t^T X_1(\tau)\, d\tau \tag{3-16}$$

$$X_3(T) - X_3(t) = F\int_t^T \frac{P_3(t) - (\tau - t)P_4(t)}{\{X_0(t) - (\tau - t)\dot{m}\}\{[P_1(t) - (\tau - t)P_2(t)]^2 + [P_3(t) - (\tau - t)P_4(t)]^2\}^{1/2}}\, d\tau - g_y^* \int_t^T d\tau$$

$$X_4(T) - X_4(t) = \int_t^T X_3(\tau)\, d\tau$$

The solutions of the integrals are

$$X_1(T) = X_1(t) - g_x^* T_{GO} + V_{ex}\{[P_1(t) - P_2(t)\tau_0]I_1 + P_2(t)I_2\}$$

$$X_2(T) = X_2(t) + X_1(t)T_{GO} - g_x^* T_{GO}^2/2 + V_{ex}[P_1(t) - P_2(t)\tau_0][I_2 - (\tau_0 - T_{GO})I_1]$$
$$+ V_{ex}P_2(t)\left[\left(\frac{b + 2cT_{GO}}{2c}\right) I_2 + \frac{1}{c}\left(\sqrt{a} - \sqrt{a + b\tau_0 + c\tau_0^2}\right)\right]$$

(3-17)

$$X_3(T) = X_3(t) - g_y^* T_{GO} + V_{ex}\{[P_3(t) - P_4(t)\tau_0]I_1 + P_4(t)I_2\}$$

$$X_4(T) = X_4(t) + X_3(t)T_{GO} - g_y^* T_{GO}^2/2 + V_{ex}[P_3(t) - P_4(t)\tau_0][I_2 - (\tau_0 - T_{GO})I_1]$$
$$+ V_{ex}P_4(t)\left[\left(\frac{b + 2cT_{GO}}{2c}\right) I_2 + \frac{1}{c}\left(\sqrt{a} - \sqrt{a + b\tau_0 + c\tau_0^2}\right)\right]$$

where

$$V_{ex} = F/\dot{m}$$

$$\tau_0 = X_0(t)/\dot{m}$$

$$T_{GO} = T - t$$

$$I_1 = \frac{1}{[a + b\tau_0 + c\tau_0^2]^{1/2}} \log_e\left[\frac{\alpha + \sqrt{1 + \alpha^2}}{\beta + \sqrt{1 + \beta^2}}\right]$$

$$I_2 = \frac{1}{c^{1/2}} \log_e\left[\frac{\gamma + \sqrt{1 + \gamma^2}}{\Delta + \sqrt{1 + \Delta^2}}\right]$$

$$\alpha = \frac{2a + b(\tau_0 + T_{GO}) + 2c\tau_0 T_{GO}}{(\tau_0 - T_{GO})(4ac - b^2)^{1/2}}$$

$$\beta = \frac{2a + b\tau_0}{\tau_0(4ac - b^2)^{1/2}}$$

$$\gamma = \frac{b + 2cT_{GO}}{(4ac - b^2)^{1/2}}$$

$$\Delta = \frac{b}{(4ac - b^2)^{1/2}}$$

$$a = P_1^2(t) + P_3^2(t)$$

$$b = -2(P_1(t)P_2(t) + P_3(t)P_4(t))$$

$$c = P_2^2(t) + P_4^2(t)$$

The integrals of the equations of motion (3-17) must be substituted into the boundary conditions (3-9) and solutions for $P_1(t)$, $P_2(t)$, $P_3(t)$, $P_4(t)$, and T_{GO} obtained. However, there are five unknown parameters and four boundary conditions. The fifth relation is obtained by using the Hamiltonian with $P_1 = 1$, i.e.

$$H = -K + \frac{F}{m_0}[1 + \hat{P}_3^2]^{1/2} - g_x^* + \hat{P}_2 X_1 - \hat{P}_3 g_y^* + \hat{P}_4 X_3 \tag{3-18}$$

$P_1 = 1$ and $P_i = \hat{P}_i$ $(i = 2, 3, 4)$ are substituted into (3-17). The boundary conditions (3-9) are solved using a modified secant iterative algorithm supplied by MSFC (see Appendix II). The adjoint variables are obtained by solving (3-18) for K together with $P_1 = 1/K$ and $P_i = \hat{P}_i/K$ $(i = 2, 3, 4)$.

g_x^* and g_y^* are obtained in the algorithm by use of the following relations

$$g_x^* = \frac{\mu}{2}\left[\frac{X_4(T)}{[X_2^2(T) + X_4(T)^2]^{3/2}} + \frac{X_4(t)}{[X_2^2(t) + X_4^2(t)]^{3/2}}\right]$$

$$g_y^* = \frac{\mu}{2}\left[\frac{X_2(T)}{[X_2^2(T) + X_4^2(T)]^{3/2}} + \frac{X_2(t)}{[X_2^2(t) + X_4^2(t)]^{3/2}}\right] \tag{3-19}$$

Thus the constant (over the remaining time - to - go) gravitational terms are taken as the average of the current and final gravity components.

3.3 Quasi Optimum Control Law

In the quasi-optimum control law the x and y components of vehicle acceleration are given by

$$\frac{F}{x_0} \frac{p_1}{(p_1^2 + p_3^2)^{1/2}} \quad \text{and} \quad \frac{F}{x_0} \frac{p_3}{(p_1^2 + p_3^2)^{1/2}}$$

in which approximate values of p_1 and p_3 are used. These approximations are given by

$$p_i = P_i + \sum_{i=0}^{6} m_{i,j} \xi_j \qquad i = 1,3 \tag{3-20}$$

From (3-7) however, ξ_1 thru ξ_4 are zero and $\xi_5 = x_5$ and $\xi_6 = x_6$; hence (3-20) becomes

$$p_1 = P_1 + m_{15}X_5 + m_{16}X_6 \tag{3-21}$$

$$p_3 = P_3 + m_{35}X_5 + m_{36}X_6 \tag{3-22}$$

Thus only m_{15}, m_{16}, m_{35}, and m_{33} in the matrix M are needed. These are calculated with the aid of (2-15). The coefficient matricies H_{XX}, . . . , H_{PP} appearing therein are found by performing the required partial differentiations on the Hamiltonian for the complete problem given by (3-6), and evaluating the result at $x = X$, i.e., for x_5 and $x_6 \equiv 0$. The results are

$$H_{XX} = \begin{bmatrix} H_{XX_{00}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{XX_{12}} & 0 & H_{XX_{14}} & 0 & 0 \\ 0 & H_{XX_{12}} & H_{XX_{22}} & H_{XX_{14}} & H_{XX_{24}} & 0 & 0 \\ 0 & 0 & H_{XX_{14}} & 0 & H_{XX_{34}} & 0 & 0 \\ 0 & 0 & H_{XX_{24}} & H_{XX_{34}} & H_{XX_{44}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$H_{XX_{00}} = \frac{2F}{X_0^3}(P_1^2 + P_3^2)^{1/2}$$

$$H_{XX_{12}} = \frac{\mu[3P_6X_4^3 + 9X_2P_5X_4^2 - 12P_6X_2^2X_4 - 6P_5X_2^3]}{[X_2^2 + X_4^2]^{7/2}}$$

$$H_{XX_{14}} = \frac{\mu[3P_5X_4^3 - 12P_6X_2X_4^2 - 12P_5X_2^2X_4 + 3P_6X_2^3]}{[X_2^2 + x_4^2]^{7/2}}$$

$$H_{XX_{22}} = \frac{\mu}{[X_2^2 + X_4^2]^{9/2}}[3(3P_5X_1 - 4P_6X_3)X_4^4 - 45(P_5X_3 + P_6X_1)X_2X_4^3$$

$$+ 9(-8P_5X_1 + 9P_6X_3)X_2^2X_4^2 + 60(P_5X_3 + P_6X_1)X_2^3X_4 + 12(2P_5X_1 - P_6X_3)X_2^4]$$

$$H_{XX_{24}} = \frac{\mu}{[X_2^2 + X_4^2]^{9/2}}[-12(P_6X_1 + P_5X_3)X_4^4 + 15(4P_6X_3 - 3P_5X_1)X_2X_4^3$$

$$+ 81(P_6X_1 + P_5X_3)X_2^2X_4^2 + 15(-3P_6X_3 + 4P_5X_1)X_2^3X_4 - 12(P_6X_1 + P_5X_3)X_2^4]$$

$$H_{XX_{34}} = \frac{\mu[-6P_6X_4^3 - 12P_5X_2X_4^2 + 9P_6X_2^2X_4 + 3P_5X_2^3]}{[X_2^2 + X_4^2]^{7/2}}$$

$$H_{XX_{44}} = \frac{\mu}{[X_2^2 + X_4^2]^{9/2}}[12(-P_5X_1 + 2P_6X_3)X_4^4 + 60(P_5X_3 + P_6X_1)X_2X_4^3$$

$$+ 9(9P_5X_1 - 8P_6X_3)X_2^2X_4^2 - 45(P_5X_3 + P_6X_1)X_2^3X_4 + 3(-4P_5X_1 + 3P_6X_3)X_2^2]$$

$$H_{PX} = H_{XP}' = \begin{bmatrix} 0 & H_{PX_{01}} & 0 & H_{PX_{03}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & H_{PX_{15}} & H_{PX_{16}} \\ 0 & 0 & 0 & 0 & 0 & H_{PX_{25}} & H_{PX_{26}} \\ 0 & 0 & 0 & 0 & 1 & H_{PX_{16}} & H_{PX_{36}} \\ 0 & 0 & 0 & 0 & 0 & H_{PX_{26}} & H_{PX_{46}} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

where

$$H_{PX_{01}} = \frac{-FP_1}{X_0^2(P_1^2 + P_3^2)^{1/2}}$$

$$H_{PX_{03}} = \frac{-FP_3}{X_0^2(P_1^2 + P_3^2)^{1/2}}$$

$$H_{PX_{15}} = \frac{\mu(2X_2^2 - X_4^2)}{(X_2^2 + X_4^2)^{5/2}}$$

$$H_{PX_{16}} = \frac{\mu(3X_2X_4)}{(X_2^2 + X_4^2)^{5/2}}$$

$$H_{PX_{25}} = \frac{\mu[3X_3X_4^3 + 9X_1X_2X_4^2 - 12X_2^2X_3X_4 - 6X_1X_2^3]}{[X_2^2 + X_4^2]^{7/2}}$$

$$H_{PX_{26}} = \frac{\mu[3X_1X_4^3 - 12X_2X_3X_4^2 - 12X_1X_2^2X_4 + 3X_2^3X_3]}{[X_2^2 + X_4^2]^{7/2}}$$

$$H_{PX_{36}} = \frac{\mu(-X_2^2 + 2X_4^2)}{[X_2^2 + X_4^2]^{5/2}}$$

$$H_{PX_{46}} = \frac{\mu[-6X_3X_4^3 - 12X_1X_2X_4^2 + 9X_2^2X_3X_4^2 + 3X_1X_2^3]}{[X_2^2 + X_4^2]^{7/2}}$$

$$H_{PP} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{FP_3^2}{X_0(P_1^2 + P_3^2)^{3/2}} & 0 & \dfrac{-FP_1P_3}{X_0(P_1^2 + P_3^2)^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{-FP_1P_3}{X_0(P_1^2 + P_3^2)^{3/2}} & 0 & \dfrac{FP_1^2}{X_0(P_1^2 + P_3^2)^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The result of substituting these matrices into (2-15) in component form are

$$-\dot{m}_{00} = 2m_{01}H_{PX_{01}} + 2m_{03}H_{PX_{03}} + m_{01}^2H_{PP_{11}} + 2m_{01}m_{03}H_{PP_{13}} + m_{03}^2H_{PP_{33}} + H_{XX_{00}} \quad (3\text{-}23\text{-}1)$$

$$\begin{aligned} -\dot{m}_{01} = {} & m_{02} + m_{05}H_{PX_{15}} + m_{06}H_{PX_{16}} + m_{11}H_{PX_{01}} + m_{13}H_{PX_{03}} + m_{11}H_{PP_{11}} \\ & + (m_{11}m_{03} + m_{13}m_{01})H_{PP_{13}} + m_{13}m_{03}H_{PP_{33}} \end{aligned} \quad (3\text{-}23\text{-}2)$$

$$\begin{aligned} -\dot{m}_{02} = {} & m_{05}H_{PX_{25}} + m_{06}H_{PX_{26}} + m_{12}H_{PX_{01}} + m_{23}H_{PX_{03}} + m_{12}m_{01}H_{PP_{11}} \\ & + (m_{12}m_{03} + m_{23}m_{01})H_{PP_{13}} + m_{23}m_{03}H_{PP_{33}} \end{aligned} \quad (3\text{-}23\text{-}3)$$

$$-\dot{m}_{03} = m_{04} + m_{05}H_{PX_{16}} + m_{06}H_{PX_{36}} + m_{13}H_{PX_{01}} + m_{33}H_{PX_{03}} + m_{13}m_{01}H_{PP_{11}}$$
$$+ (m_{13}m_{03} + m_{33}m_{01})H_{PP_{13}} + m_{03}m_{33}H_{PP_{33}} \tag{3-23-4}$$

$$-\dot{m}_{04} = m_{05}H_{PX_{26}} + m_{06}H_{PX_{46}} + m_{14}H_{PX_{01}} + m_{34}H_{PX_{03}} + m_{14}m_{01}H_{PP_{11}}$$
$$+ (m_{14}m_{03} + m_{34}m_{01})H_{PP_{13}} + m_{34}m_{03}H_{PP_{33}} \tag{3-23-5}$$

$$-\dot{m}_{05} = m_{01} + m_{15}H_{PX_{01}} + m_{35}H_{PX_{03}} + m_{15}m_{01}H_{PP_{11}}$$
$$+ (m_{15}m_{03} + m_{35}m_{01})H_{PP_{13}} + m_{35}m_{03}H_{PP_{33}} \tag{3-23-6}$$

$$-\dot{m}_{06} = m_{03} + m_{16}H_{PX_{01}} + m_{36}H_{PX_{03}} + m_{16}m_{01}H_{PP_{11}}$$
$$+ (m_{16}m_{03} + m_{36}m_{01})H_{PP_{13}} + m_{36}m_{03}H_{PP_{33}} \tag{3-23-7}$$

$$-\dot{m}_{11} = 2m_{12} + 2m_{15}H_{PX_{15}} + 2m_{16}H_{PX_{16}} + m_{11}^2H_{PP_{11}} + 2m_{11}m_{13}H_{PP_{13}} + m_{13}^2H_{PP_{33}} \tag{3-23-8}$$

$$-\dot{m}_{12} = m_{15}H_{PX_{25}} + m_{16}H_{PX_{26}} + m_{22} + m_{25}H_{PX_{15}} + m_{26}H_{PX_{16}} + m_{11}m_{12}H_{PP_{11}}$$
$$+ (m_{12}m_{13} + m_{11}m_{23})H_{PP_{13}} + m_{13}m_{23}H_{PP_{33}} + H_{XX_{12}} \tag{3-23-9}$$

$$-\dot{m}_{13} = m_{14} + m_{23} + m_{15}H_{PX_{16}} + m_{16}H_{PX_{36}} + m_{35}H_{PX_{15}} m_{36}H_{PX_{16}} + m_{11}m_{13}H_{PP_{11}}$$
$$+ (m_{13}^2 + m_{11}m_{13})H_{PP_{13}} + m_{13}m_{33}H_{PP_{33}} \tag{3-23-10}$$

$$-\dot{m}_{14} = m_{24} + m_{15}H_{PX_{26}} + m_{16}H_{PX_{46}} + m_{45}H_{PX_{15}} + m_{46}H_{PX_{16}} + m_{11}m_{14}H_{PP_{11}}$$
$$+ (m_{13}m_{14} + m_{11}m_{34})H_{PP_{13}} + m_{13}m_{34}H_{PP_{33}} + H_{XX_{14}} \tag{3-23-11}$$

$$-\dot{m}_{15} = m_{11} + m_{25} + m_{55}H_{PX_{15}} + m_{56}H_{PX_{16}} + m_{11}m_{15}H_{PP_{11}}$$
$$+ (m_{13}m_{15} + m_{11}m_{35})H_{PP_{13}} + m_{13}m_{35}H_{PP_{33}} \qquad (3\text{-}23\text{-}12)$$

$$-\dot{m}_{16} = m_{13} + m_{26} + m_{56}H_{PX_{15}} + m_{66}H_{PX_{16}} + m_{11}m_{16}H_{PP_{11}}$$
$$+ (m_{13}m_{16} + m_{11}M_{36})H_{PP_{13}} + m_{13}m_{36}H_{PP_{33}} \qquad (3\text{-}23\text{-}13)$$

$$-\dot{m}_{22} = m_{25}H_{PX_{25}} + 2m_{26}H_{PX_{26}} + m_{12}^{2}H_{PP_{11}} + 2m_{12}m_{23}H_{PP_{13}} + m_{23}^{2}H_{PP_{33}} + H_{XX_{22}} \qquad (3\text{-}23\text{-}14)$$

$$-\dot{m}_{23} = m_{24} + m_{25}H_{PX_{16}} + m_{26}H_{PX_{36}} + m_{35}H_{PX_{25}} + m_{36}H_{PX_{26}} + m_{12}m_{13}H_{PP_{11}}$$
$$+ (m_{13}m_{23} + m_{12}m_{33})H_{PP_{13}} + m_{23}m_{33}H_{pp_{33}} + H_{XX_{14}} \qquad (3\text{-}23\text{-}15)$$

$$-\dot{m}_{24} = m_{25}H_{PX_{26}} + m_{26}H_{PX_{46}} + m_{45}H_{PX_{25}} + m_{46}H_{PX_{26}} + m_{12}m_{14}H_{PP_{11}}$$
$$+ (m_{14}m_{23} + m_{34}m_{12})H_{PP_{13}} + m_{34}m_{23}H_{PP_{33}} + H_{XX_{24}} \qquad (3\text{-}23\text{-}16)$$

$$-\dot{m}_{25} = m_{12} + m_{55}H_{PX_{25}} + m_{56}H_{PX_{26}} + m_{12}m_{15}H_{PP_{11}} + (m_{15}m_{23} + m_{12}m_{35})H_{PP_{13}}$$
$$+ m_{23}m_{35}H_{PP_{33}} \qquad (3\text{-}23\text{-}17)$$

$$-\dot{m}_{26} = m_{23} + m_{56}H_{PX_{25}} + m_{66}H_{PX_{26}} + m_{12}m_{16}H_{PP_{11}} + (m_{16}m_{23} + m_{12}m_{36})H_{PP_{13}}$$
$$+ m_{23}m_{36}H_{PP_{33}} \qquad (3\text{-}23\text{-}18)$$

$$-\dot{m}_{33} = 2m_{34} + 2m_{35}H_{PX_{16}} + 2m_{36}H_{PX_{36}} + m_{13}^{2}H_{PP_{11}} + 2m_{13}m_{33}H_{PP_{13}} + m_{33}^{2}H_{PP_{33}} \qquad (3\text{-}23\text{-}19)$$

$$-\dot{m}_{34} = m_{44} + m_{35}H_{PX_{26}} + m_{36}H_{PX_{46}} + m_{45}H_{PX_{16}} + m_{46}H_{PX_{36}} + m_{13}m_{14}H_{PP_{11}} + (m_{14}m_{33} + m_{13}m_{34})H_{PP_{13}} + m_{33}m_{34}H_{PP_{33}} + H_{XX_{34}} \tag{3-23-20}$$

$$-\dot{m}_{35} = m_{13} + m_{45} + m_{55}H_{PX_{16}} + m_{56}H_{PX_{36}} + m_{13}m_{15}H_{PP_{11}} + (m_{15}m_{33} + m_{35}m_{13})H_{PP_{13}} + m_{35}m_{33}H_{PP_{33}} \tag{3-23-21}$$

$$-\dot{m}_{36} = m_{33} + m_{46} + m_{56}H_{PX_{16}} + m_{66}H_{PX_{36}} + m_{16}m_{13}H_{PP_{11}} + (m_{16}m_{33} + m_{36}m_{13})H_{PP_{13}} + m_{36}m_{33}H_{PP_{33}} \tag{3-23-22}$$

$$-\dot{m}_{44} = 2m_{45}H_{PX_{26}} + 2m_{46}H_{PX_{46}} + m_{14}^{2}H_{PP_{11}} + 2m_{14}m_{34}H_{PP_{13}} + m_{34}^{2}H_{PP_{33}} + H_{XX_{44}} \tag{3-23-23}$$

$$-\dot{m}_{45} = m_{14} + m_{55}H_{PX_{26}} + m_{56}H_{PX_{46}} + m_{14}m_{15}H_{PP_{11}} + (m_{15}m_{34} + m_{34} + m_{14}m_{35})H_{PP_{13}} + m_{34}m_{35}H_{PP_{33}} \tag{3-23-24}$$

$$-\dot{m}_{46} = m_{34} + m_{56}H_{PX_{26}} + m_{66}H_{PX_{46}} + m_{14}m_{16}H_{PP_{11}} + (m_{16}m_{34} + m_{14}m_{36})H_{PP_{13}} + m_{34}m_{36}H_{PP_{33}} \tag{3-23-25}$$

$$-\dot{m}_{55} = 2m_{15} + m_{15}^{2}H_{PP_{11}} + 2m_{15}m_{35}H_{PP_{13}} + m_{35}^{2}H_{PP_{33}} \tag{3-23-26}$$

$$-\dot{m}_{56} = m_{35} + m_{16} + m_{15}m_{16}H_{PP_{11}} + (m_{15}m_{36} + m_{16}m_{35})H_{PP_{13}} + m_{35}m_{36}H_{PP_{33}} \tag{3-23-27}$$

$$-\dot{m}_{66} = 2m_{36} + m_{16}^{2}H_{PP_{11}} + 2m_{16}m_{36}H_{PP_{13}} + m_{36}^{2}H_{PP_{33}} \tag{3-23-28}$$

Note that equations (3-23-26, 27, 28) with $\dot{m}_{55}$, $\dot{m}_{56}$, and $\dot{m}_{66}$ equal to zero involve only the four required correction coefficients m_{15} , m_{16} , m_{35} and m_{36} .

$$0 = 2m_{15} + m_{15}^2 H_{PP_{11}} + 2m_{15} m_{35} H_{PP_{13}} + m_{35}^2 H_{PP_{33}} \tag{3-24a}$$

$$0 = m_{35} + m_{16} + m_{15} m_{16} H_{PP_{11}} + (m_{15} m_{36} + m_{16} m_{35}) H_{PP_{13}} + m_{35} m_{36} H_{PP_{33}} \tag{3-24b}$$

$$0 = 2m_{36} + m_{16}^2 H_{PP_{11}} + 2m_{16} m_{36} H_{PP_{13}} + m_{36}^2 H_{PP_{33}} \tag{3-24c}$$

Since $H_{PP_{11}} H_{PP_{33}} = H_{PP_{13}}^2$ equations (3-24) become

$$0 = 2m_{15} + (m_{15} H_{PP_{11}}^{1/2} - m_{35} H_{PP_{13}}^{1/2})^2 \tag{3-25a}$$

$$0 = m_{35} + m_{16} (m_{15} H_{PP_{11}}^{1/2} - m_{35} H_{PP_{33}}^{1/2})(m_{16} H_{PP_{11}}^{1/2} - m_{36} H_{PP_{33}}^{1/2}) \tag{3-25b}$$

$$0 = 2m_{36} + (m_{16} H_{PP_{11}}^{1/2} - m_{36} H_{PP_{33}}^{1/2})^2 \tag{3-25c}$$

A fourth equation is obtained by solving (3-23-6) and (3-23-7) with $\dot{m}_{05}$ and $\dot{m}_{06}$ equal to zero for m_{01} and m_{03} as a function of m_{15}, m_{16}, m_{35} and m_{36} . These relations are

$$\begin{aligned} m_{01} = &\{1 + m_{16} H_{PP_{13}} + m_{36} H_{PP_{33}}\}\{m_{15} H_{PX_{01}} + m_{35} H_{PX_{03}}\}/D \\ &+ \{m_{15} H_{PP_{13}} + m_{35} H_{PP_{33}}\}\{m_{16} H_{PX_{01}} + m_{36} H_{PX_{03}}\}/D \end{aligned} \tag{3-26}$$

$$\begin{aligned} m_{03} = &\{m_{16} H_{PP_{11}} + m_{36} H_{PP_{13}}\}\{m_{15} H_{PX_{01}} + m_{35} H_{PX_{03}}\}/D \\ &- \{1 + m_{15} H_{PP_{11}} + m_{35} H_{PP_{13}}\}\{m_{16} H_{PX_{01}} + m_{36} H_{PX_{03}}\}/D \end{aligned} \tag{3-27}$$

where

$$D = 1 + m_{15} H_{PP_{11}} + (m_{16} + m_{35}) H_{PP_{13}} + m_{36} H_{PP_{33}} \tag{3-28}$$

Substituting (3-26), (3-27), and (3-28) into (3-23-1) with $\dot{m}_{00}$ equal zero and simplifying results in

$$0 = \frac{F}{X_0^3(P_1^3 + P_3^2)^{1/2}} [-2X_0(m_{01}P_1 + m_{03}P_3) + \frac{X_0^2}{(P_1^2 + P_3^2)^2} (m_{01}P_3 - m_{03}P_1)^2 + (P_1^2 + P_3^2)] \tag{3-29}$$

Note that the presence of $H_{XX_{00}}$ in (3-23-1) eliminates the trivial solution $m_{15} = m_{16} = m_{35} = m_{36} = 0$.

Simplifying (3-25a,b,c) results in

$$0 = 2m_{15} + \frac{F}{X_0(P_1^2 + P_3^2)^{3/2}} (m_{15}P_3 - m_{35}P_1)^2 \tag{3-30}$$

$$0 = m_{35} + m_{16} + \frac{F}{X_0(P_1^2 + P_3^2)^{3/2}} (m_{15}P_3 - m_{35}P_1)(m_{16}P_3 - m_{36}P_1) \tag{3-31}$$

$$0 = 2m_{36} + \frac{F}{X_o(P_1^2 + P_3^2)^{3/2}} (m_{16}P_3 - m_{36}P_1)^2 \tag{3-32}$$

Equations (3-29) thru (3-32) are solved simultaneously using the modified secant algorithm to obtain the required correction coefficients.

3.4 Comparison of Guidance Law Performance

This section quantitatively describes the performance of four explicit guidance techniques used to inject a payload into a fixed orbit with minimum fuel expenditure. The booster is assumed to have a constant thrust magnitude. Some of the difficulties and limitations inherent in each guidance technique are examined. The four guidance laws examined are:

(1) Iterative Guidance Mode (IGM)

(2) Quasi-Optimum

(3) Simplified

(4) Polynomial Gravity Expansion (PGE)

The IGM (6,7,8), developed by MSFC, is an explicit guidance law which approximates the central gravitational field by a uniform field over a flat earth. Applying the equations derived from the simplified model to a spherical earth and an initial estimate of the time-to-go, the terminal range angle is computed as a function of the current position and the additional displacement that would be obtained during the remaining time-to-go. Since the predicted range angle may be in error, a weighting factor may be used in this computation to account for gravity loss on steep and/or long duration trajectories. The computation used to update the time-to-go is derived by simultaneously solving the velocity to be gained equation at a given instant of time together with the characteristic velocity equation.

The steering angle is computed in two parts. The first part is assumed to be the major portion of the steering angle and is a constant angle obtained by utilizing the optimum steering constraints required to obtain a given velocity without position constraints over a flat earth. The second smaller part is comprised of a linear time function which is added to the first part and substituted into the equations of motion. A small angle approximation is made and the equations of motion are integrated and relations obtained, subject to the optimality constraints for the flat earth model, to obtain the complete steering angle. The computer flow diagram for the IGM algorithm used in this investigation is provided in Appendix I.

The simplified guidance law derived in Section 3.2 utilizes a modified secant algorithm for n equations in n unknowns to solve (3-17) for P_1, P_2, P_3, P_4 and time-to-go. The vehicle is steered according to

$$\tan \chi = \frac{P_3}{P_1} \qquad (3\text{-}33)$$

A description of the modified secant algorithm is provided in Appendix II.

The quasi optimum guidance law derived in Section 3.3 is exactly the same as the simplified law with the additional step of employing the secant algorithm a second time to solve equations (3-29) through (3-33) for m_{15}, m_{16}, m_{35}, and m_{36} and steers the vehicle according to

$$\tan \chi = \frac{P_3 + m_{35}x_5 + m_{36}x_6}{P_1 + m_{15}x_5 + m_{16}x_6}$$

The PGE explicit guidance law which is currently under investigation at MSFC is similar to the simplified guidance law with the exception of the gravitational model used in the equations of motion. The simplified law utilized (3-14) to derive the constant (over the remaining time-to-go) gravity terms which are an average of the current and predicted final gravity terms. The PGE uses a third order polynomial to approximate the gravity components, i.e.,

$$g_i = \mu \sum_{j=1}^{4} C_{ij} T_{GO}^{\,j-1} \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2, 3, 4 \end{array} \qquad (3\text{-}34)$$

The algorithm used to calculate the C_{ij}'s was furnished by MSFC and is presented in Appendix III.

Three sets of optimum trajectories were generated by integrating the exact equations of motion with aptimum steering backwards in time from a 100 nautical mile circular orbit about the earth. The burnout state and adjoint vectors for each run were obtained from the circular orbit boundary conditions (3-3) and the variational Hamiltonian (3-6). The four guidance laws were then used to steer the vehicle for each set of initial conditions obtained from the optimum trajectories. In each case the guidance law computation was performed every second of flight time. The computation of the adjoint variables was stopped when $T_{GO} < 10$ seconds and during the remainder of the flight the vehicle was steered with the last set of adjoint variables although the calculation of T_{GO} was continued until $T_{GO} < 1$ second. The engine was shut down when $T_{GO} = 0$.

The vehicle thrust was taken as 889,646 Newtons (200,000 pounds) and mass flow rate as 204.3146 kgm/second (14 slugs/second) for all runs. The parameters common to each set of trajectories are listed in Table 3-1

Table 3-1

Parameters for Three Sets of Optimum Trajectories

Parameter	Set 1	Set 2	Set 3
Burn Time (Seconds)	200	500	1000
Ignition Mass (Kilograms)	151614.12	218908.5	934009.6
Burnout Mass (Kilograms)	116751.2	116751.2	729695.0

On relatively flat 200 second trajectories which are typical of Saturn third stage flights all four guidance laws performed extremely well. The engine burn times were between 200.05 and 200.1 seconds for each run. Figures 3-2 through 3-4 illustrate the position, velocity, and pitch angle for a steep optimum 200 second trajectory. On this trajectory the IGM required a large weighting factor to obtain a reasonable initial pitch angle and did not perform as well as the other guidance laws. Table 3-2 lists the engine burn times required to insert the booster into a 100 nautical mile orbit from the ignition point illustrated in Figure 3-2.

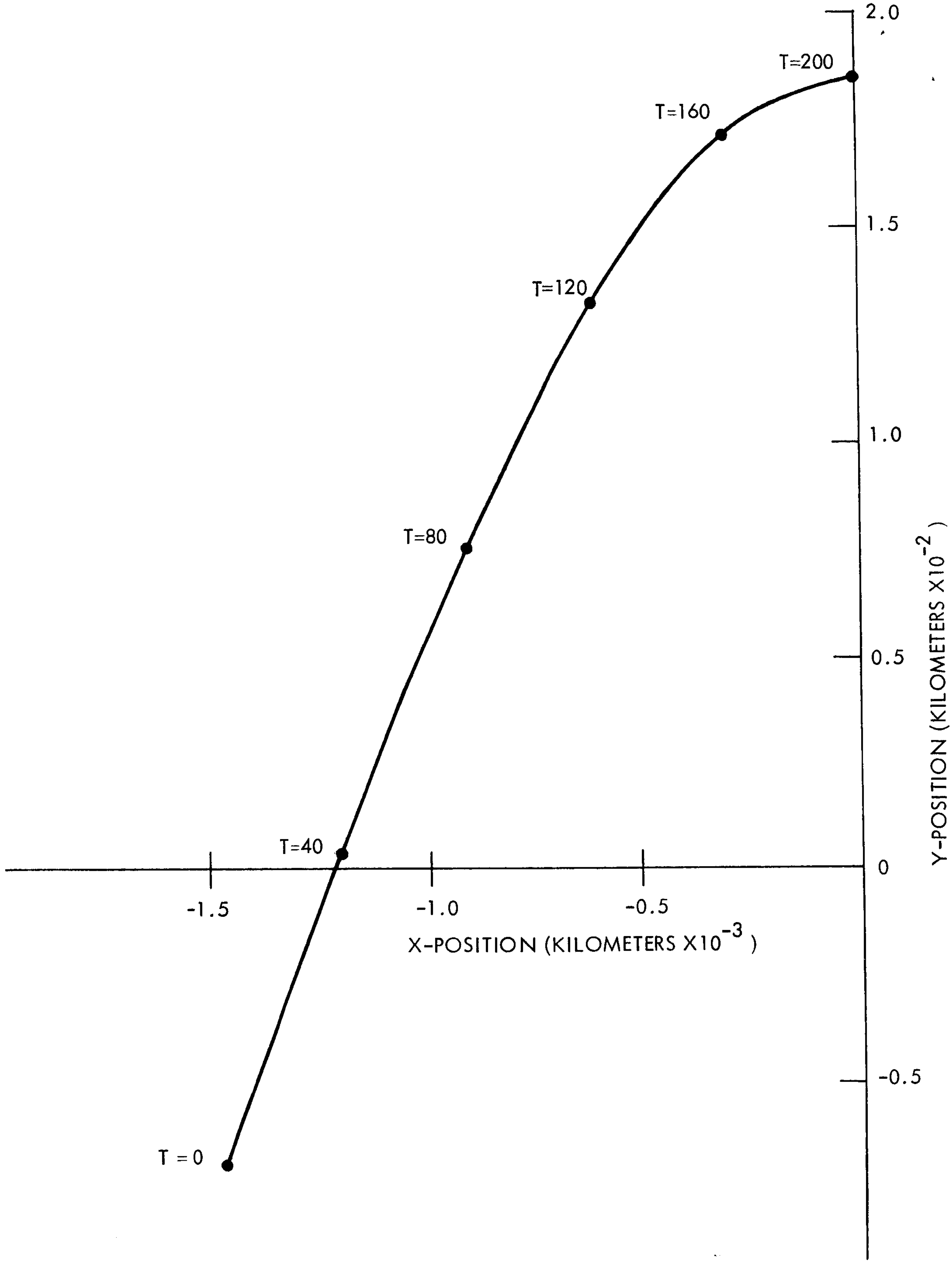
Table 3-2

Burn Times Required to Achieve Orbit for a 200 Second Optimum Trajectory

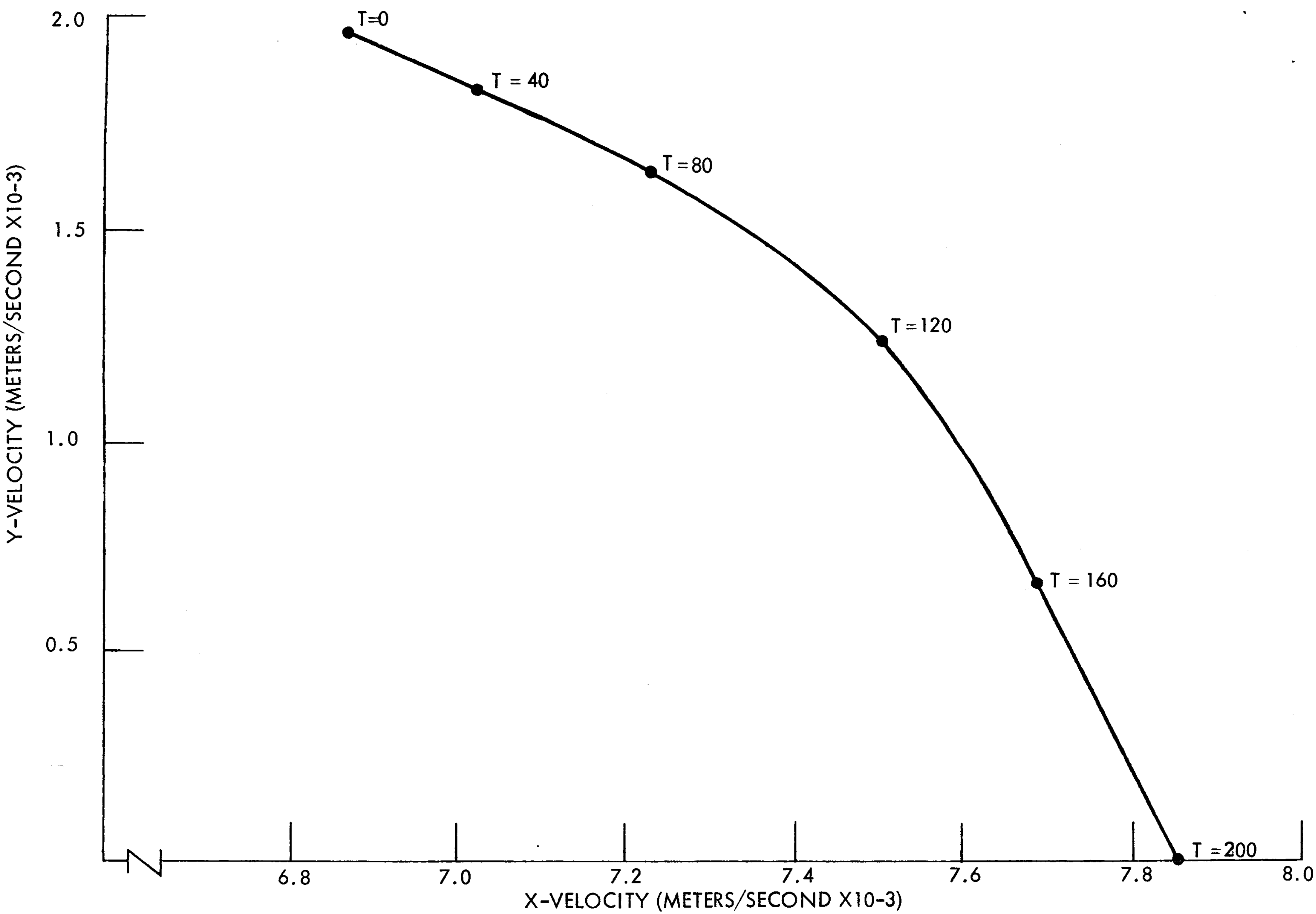
Guidance Law	Burn Time (Seconds)
Quasi-Optimum	200.74
Simplified	200.77
PGE	200.81
IGM	237.89

A 500 and 1000 second optimum trajectory were generated which are not typical of current Saturn requirements but may well represent future mission requirements. The position, velocity, and pitch angles for these trajectories are illustrated in Figures 3-5 through 3-10. The IGM required a weighting factor to account for the long duration of the 500 second trajectory but could not determine a reasonable pitch angle for the 1000 second trajectory and failed to achieve the desired orbit. The PGE guidance law performed very well on the 500 second trajectory but the algorithm to determine the steering angle did not converge on the 1000 second trajectory and the vehicle failed to achieve orbit. The quasi-optimum and simplified guidance laws achieved the required orbits on both trajectories with excellent burn times on the 500 second run, but the simplified law gave poor results on the 1000 second run. The quasi-optimum control law, however, achieves, the desired orbit in 1001.56 seconds!

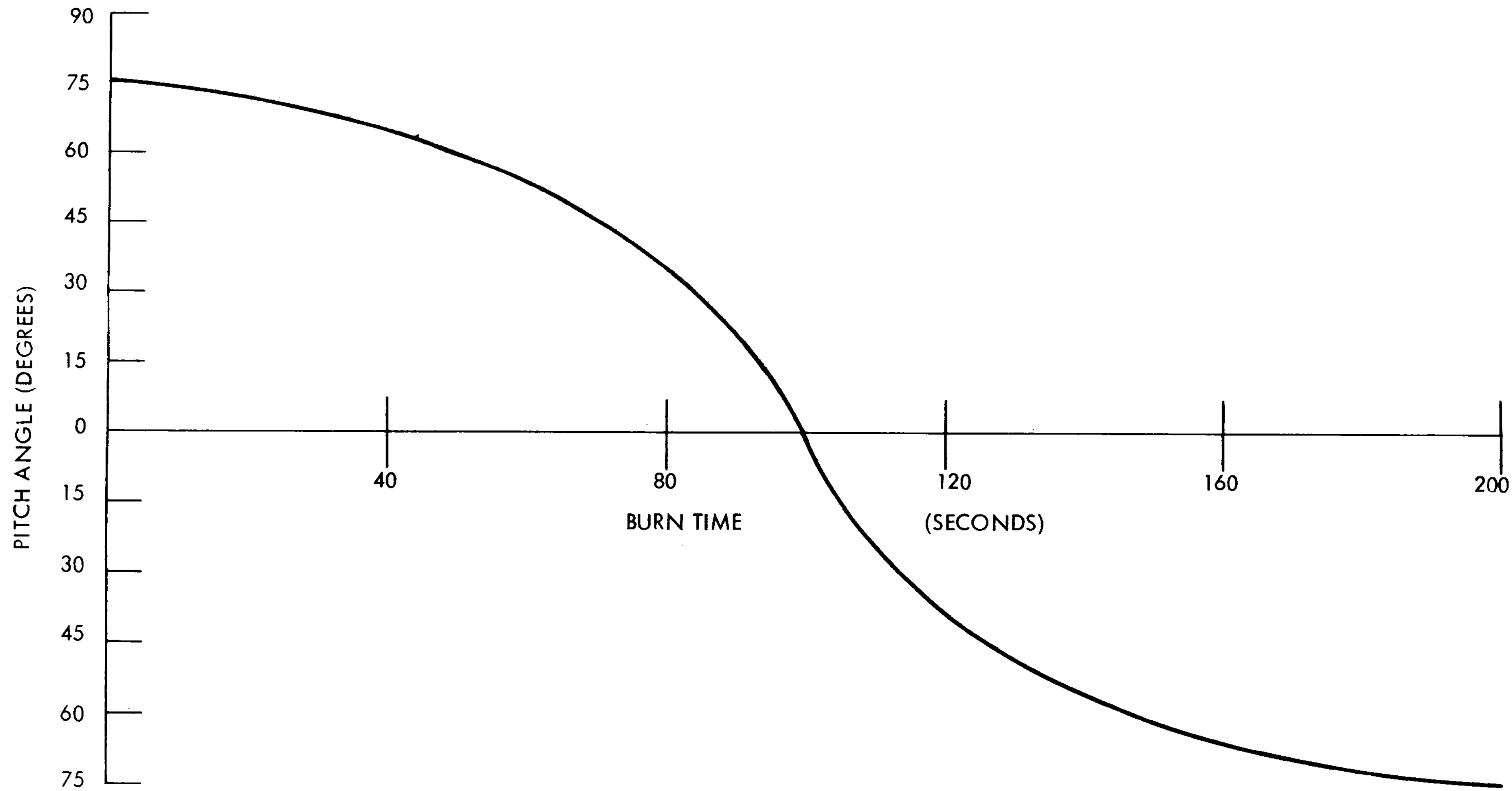
Table 3-3 lists the engine burn times required to achieve the orbital conditions for the 500 and 1000 second trajectories.



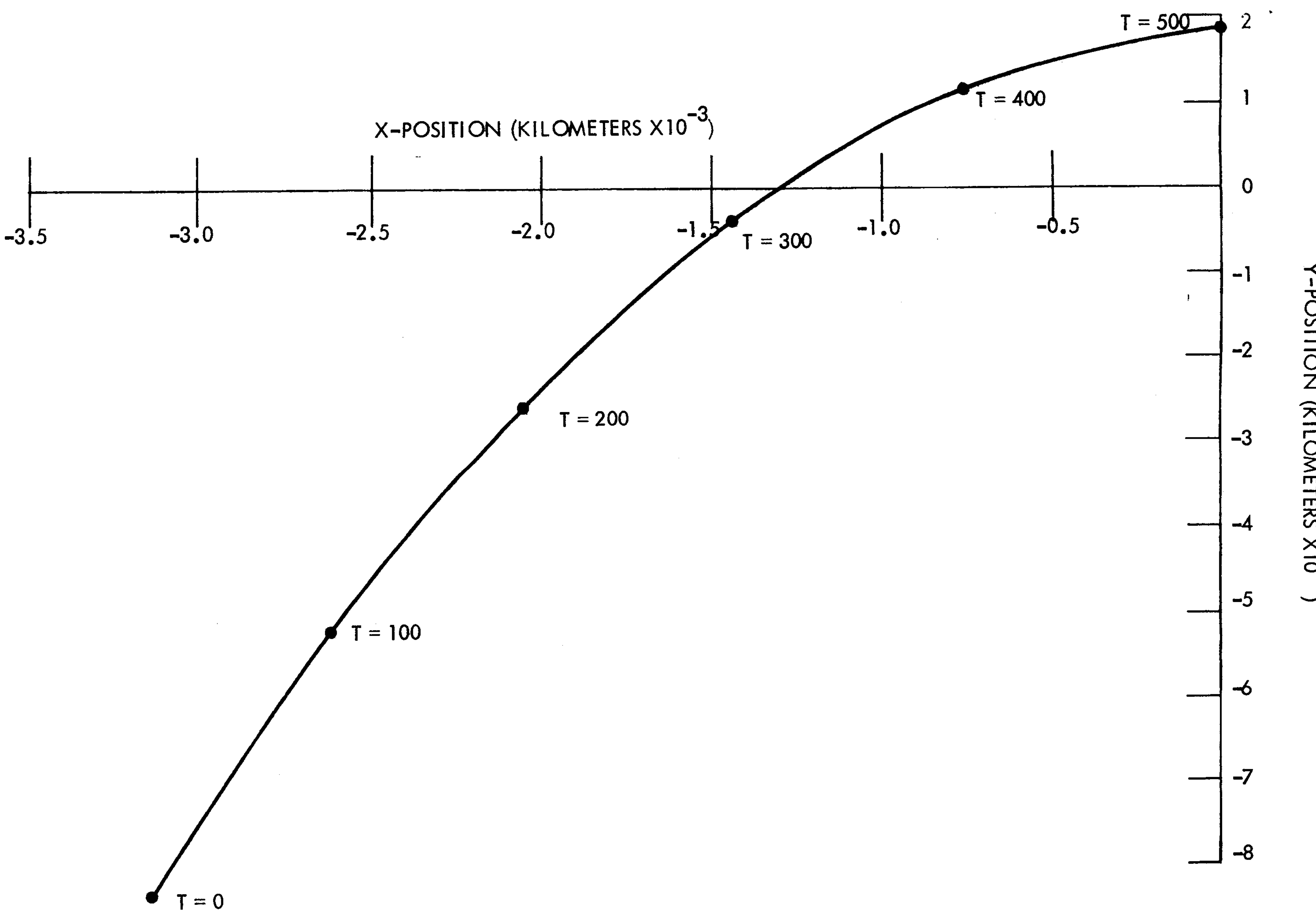
X VS Y POSITION - 200 SECOND OPTIMUM TRAJECTORY
FIGURE 3-2



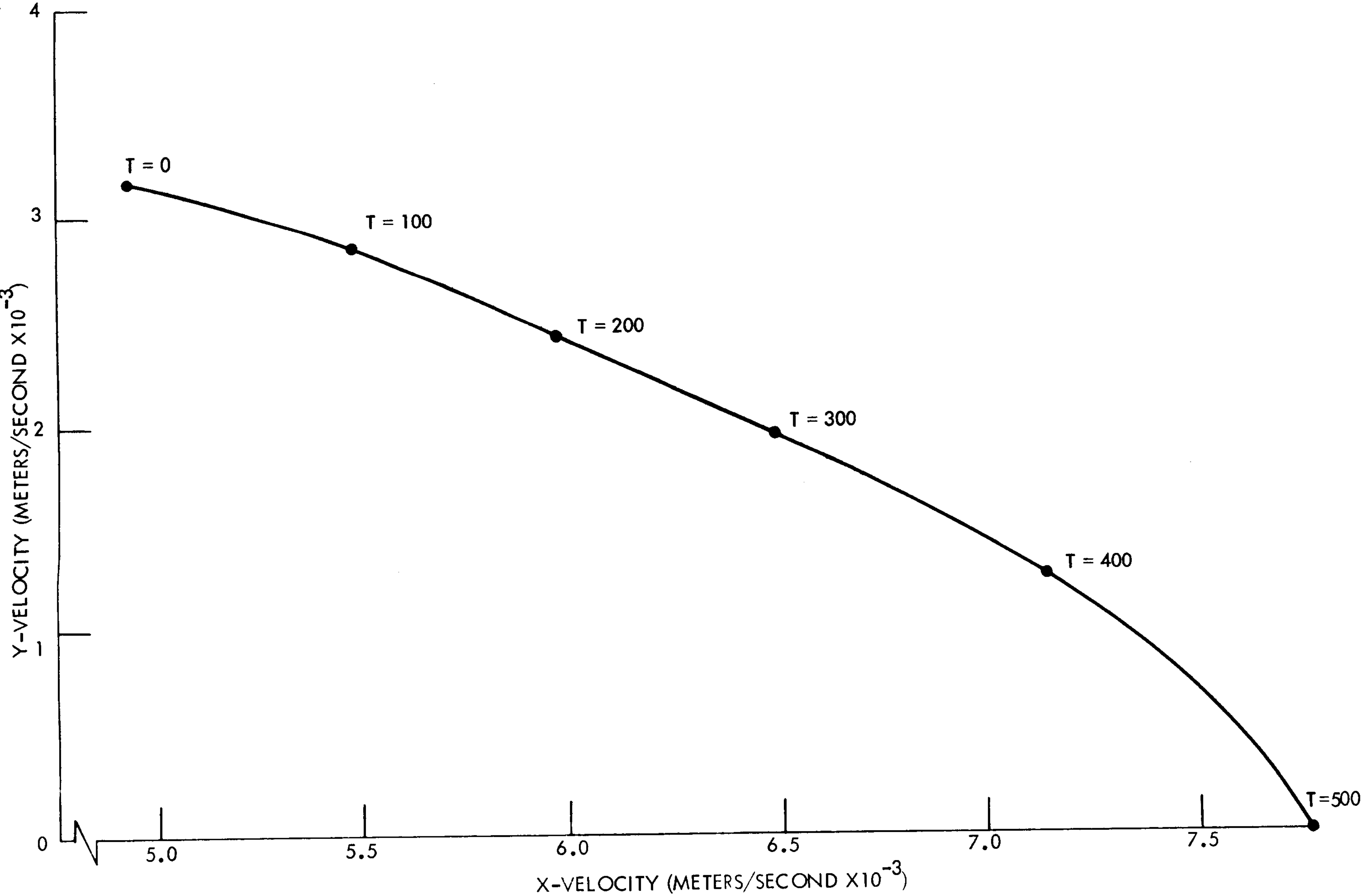
X VS Y VELOCITY - 200 SECOND OPTIMUM TRAJECTORY
FIGURE 3-3



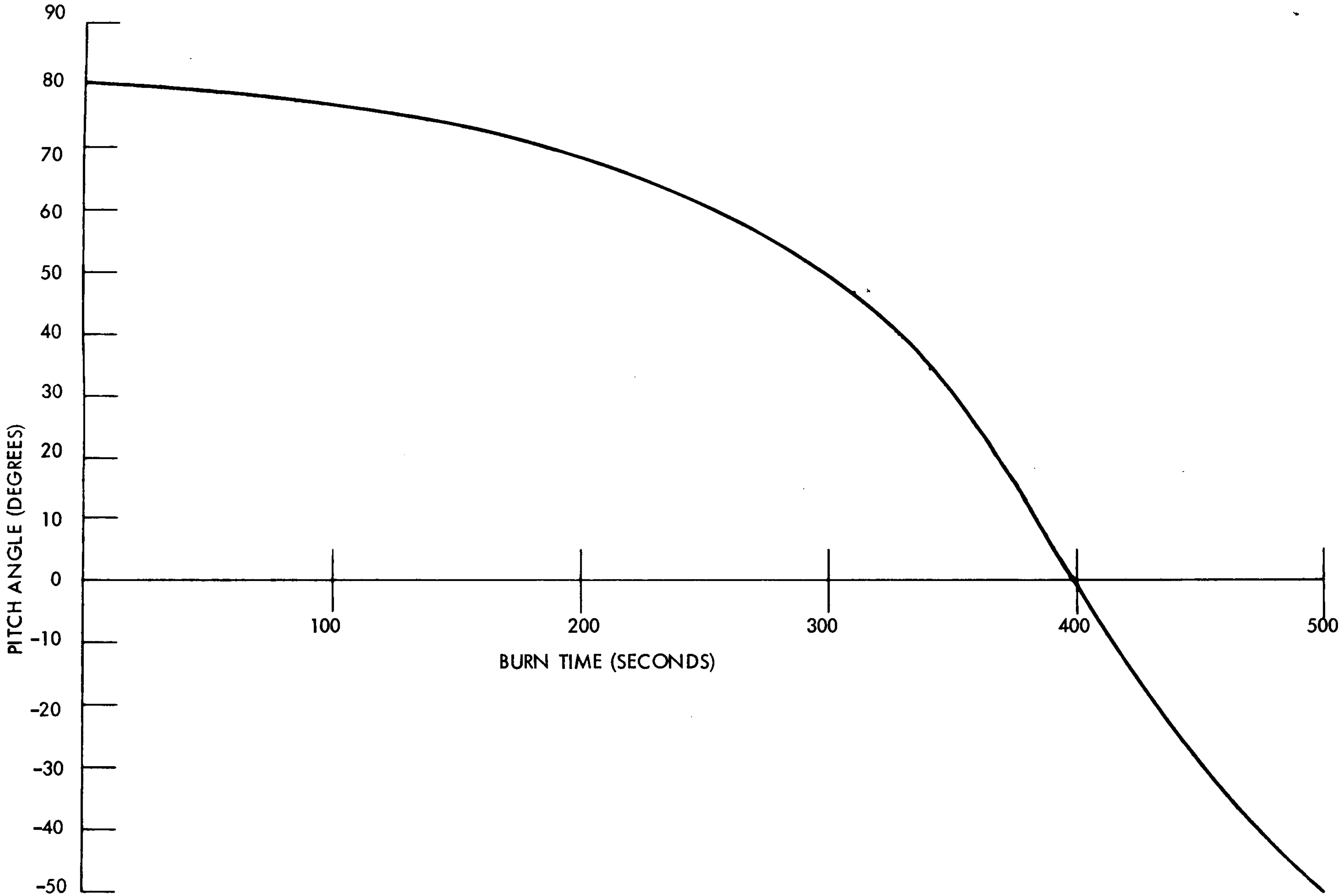
PITCH ANGLE VS. TIME - 200 SECOND OPTIMUM TRAJECTORY
FIGURE 3-4



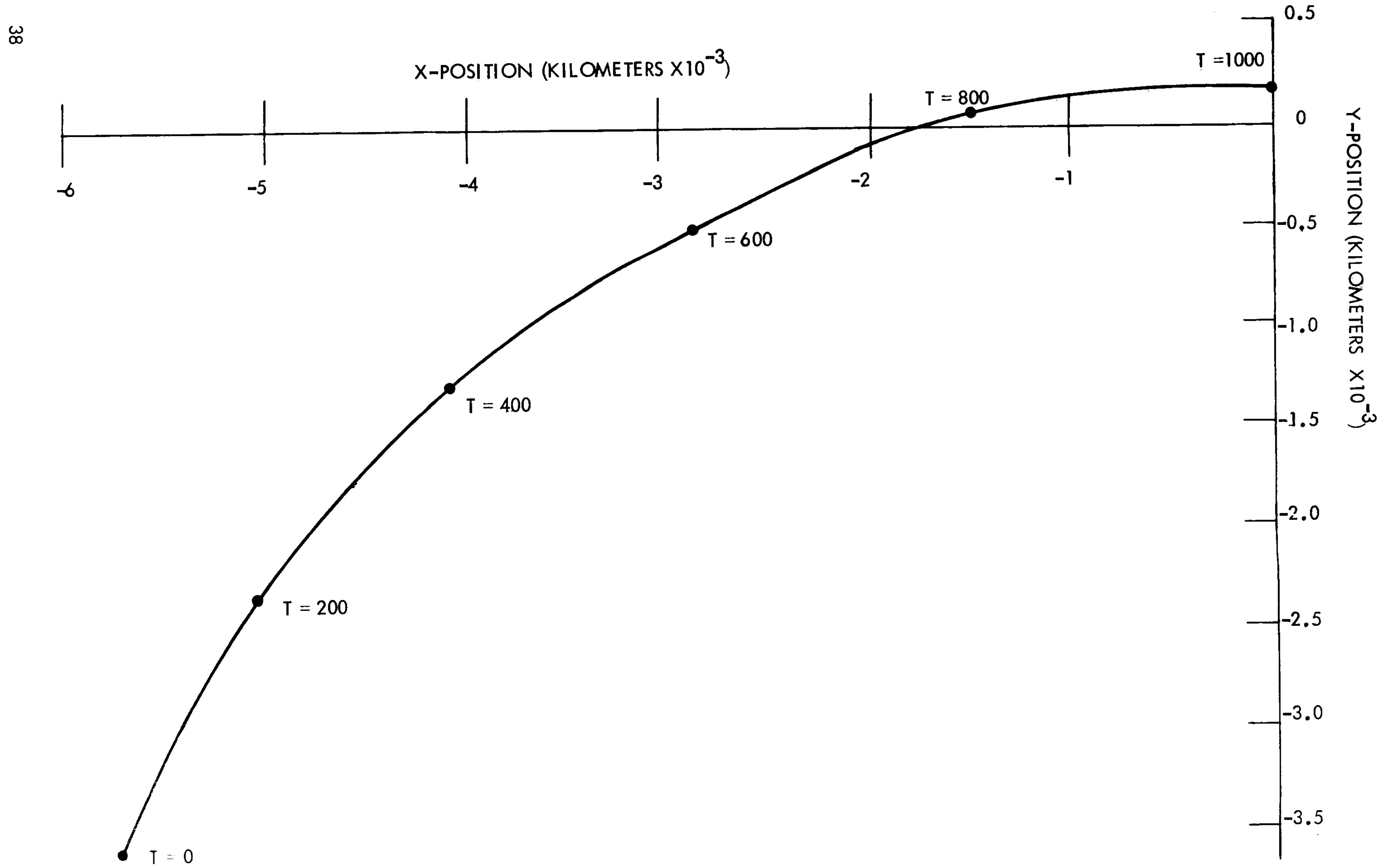
X VS. Y POSITION - 500 SECOND OPTIMUM TRAJECTORY
FIGURE 3-5



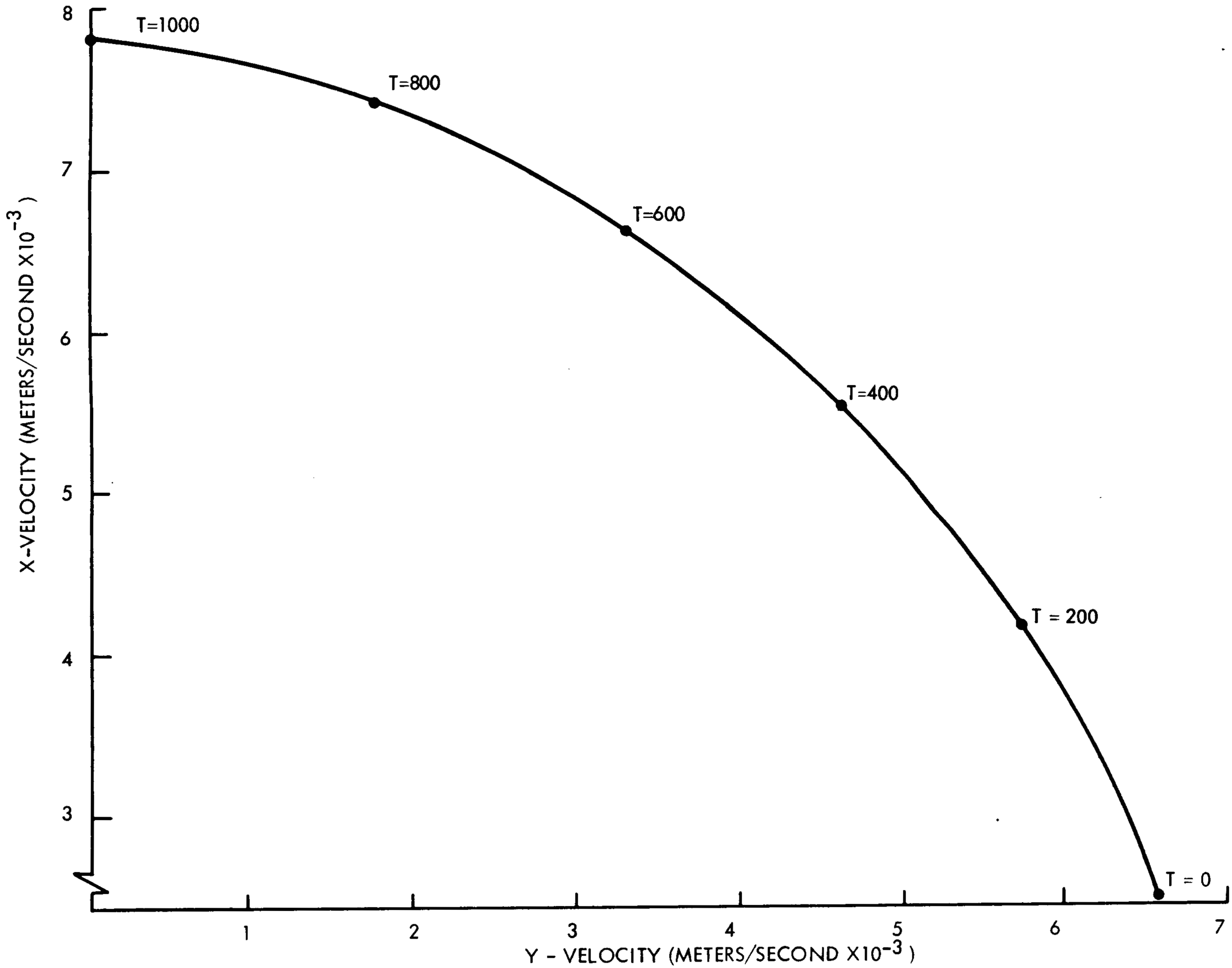
X VS. Y VELOCITY - 500 SECOND OPTIMUM TRAJECTORY
FIGURE 3-6



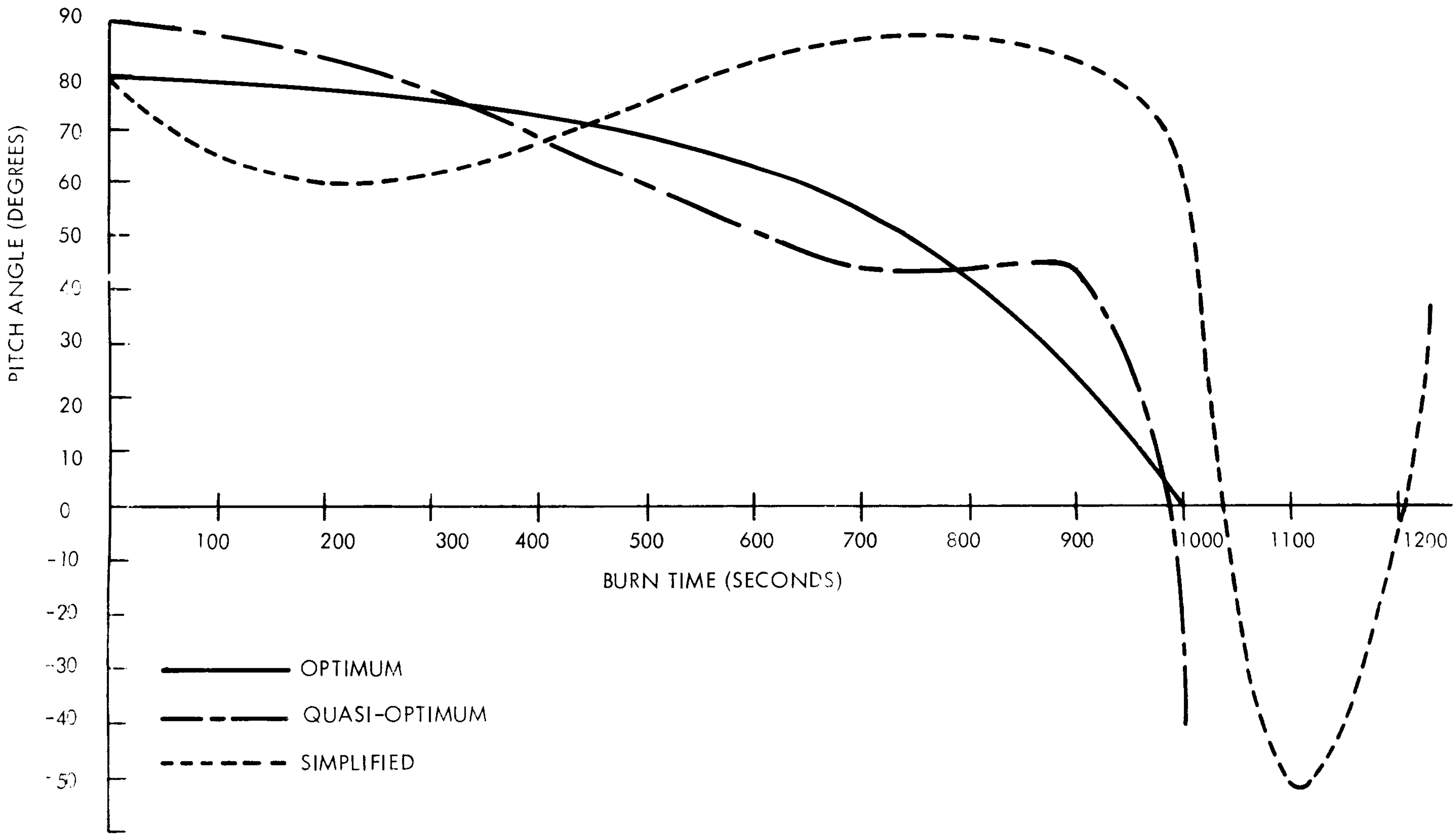
PITCH ANGLE VS. TIME - 500 SECOND OPTIMUM TRAJECTORY
FIGURE 3-7



X VS Y POSITION - 1000 SECOND OPTIMUM TRAJECTORY
FIGURE 3-8



X VS Y VELOCITY - 1000 SECOND OPTIMUM TRAJECTORY
FIGURE 3-9



PITCH ANGLE FOR OPTIMUM, QUASI-OPTIMUM, AND SIMPLIFIED
1000 SECOND TRAJECTORIES
FIGURE 3-10

Table 3-3

Burn Times Required to Achieve Orbit for 500 and 1000 Second Optimum Trajectories

Guidance Law	Burn Time (Seconds)	
	500	1000
IGM	539.29	Orbit not obtained
Quasi Optimum	500.53	1001.56
Simplified	500.60	1231.28
PGE	501.14	Orbit not obtained

Figure 3-10 illustrates the pitch angle vs. time for the 1000 second optimum trajectory. The pitch angle commands generated by the simplified and quasi-optimum guidance laws are also given.

4. APPLICATION TO LOW-THRUST GUIDANCE

4.1 Equation of Motion

The primary purpose of the investigation was to develop and evaluate a feedback guidance law using the quasi-optimum control technique. Therefore for simplicity, but without loss in generality, a planar trajectory was considered. The state variables chosen to represent the motion of the vehicle are a hybrid set of orbital elements in which the zero-eccentricity singularity has been removed. The independent variable is τ , where τ is related to real-time t by $d\tau = n\,dt$, n being the "mean motion".

At first glance it would appear that the system can be adequately represented by a 5-component state vector; 4 for the orbital (planar) motion and 1 for the mass of the vehicle. However, the dynamic equations contain the independent variable explicitly and, therefore, are not compatible with the quasi-optimum control theory which requires that the system be autonomous. The system can easily be made autonomous by including the independent variable as a separate component of the state vector. The following six component state vector is used to completely specify the state of the system

$$x_1 = a/a_0$$

$$x_2 = e \sin \omega$$

$$x_3 = e \cos \omega$$

$$x_4 = t$$

$$x_5 = m/m_0$$

$$x_6 = \tau$$

where

a = semi major axis

$e = (x_2^2 + x_3^2)^{\frac{1}{2}}$ = eccentricity

$\omega = \tan^{-1}(x_2/x_3)$ = argument of perigee

t = time

m = instantaneous mass

τ = non-dimensional time = $\int n\,dt$

$n = (\mu/a^3)^{\frac{1}{2}}$ = mean motion

μ = gravitational constant of the attracting body

Subscript 0 = reference values

Other terms which will be used subsequently are defined as

$T = \text{thrust}/m_0$ = reference thrust acceleration (T/x_5 = instantaneous thrust acceleration)

α = thrust angle measured with respect to local horizontal

c = exhaust velocity of propellant = $I_{sp}g$

$k = 1/c$

The equation of motion in terms of the above six-component state vector is given by

$$
\begin{aligned}
\frac{dx}{d\tau} &= F(x)\, u \\
\frac{dx_4}{d\tau} &= (x_1 a_0)^{3/2}/\mu^{\frac{1}{2}} \\
\frac{dx_5}{d\tau} &= -kT(x_1 a_0)^{3/2}/\mu^{\frac{1}{2}} \\
\frac{dx_6}{d\tau} &= 1
\end{aligned}
\qquad (4\text{-}1)
$$

where

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ; \qquad u \equiv \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} T\cos\alpha \\ T\sin\alpha \end{bmatrix}$$

$$F \equiv \frac{a_0^2 x_1^2}{\mu x_5} \begin{bmatrix} 2x_1\delta/\gamma & 2x_1(x_3 \sin\tau - x_2\cos\tau)/\gamma \\ \gamma(x_2 + (1+\delta)\sin\tau)/\delta & -\gamma\cos\tau \\ \gamma(x_3 + (1+\delta)\cos\tau)/\delta & \gamma\sin\tau \end{bmatrix}$$

$$k \equiv \frac{1}{c}, \quad T \equiv (u_1^2 + u_2^2)^{\frac{1}{2}} = \frac{\text{Thrust}}{m_0}$$

$$\delta \equiv 1 + x_3\cos\tau + x_2\sin\tau \quad \text{and} \quad \gamma \equiv (1 - x_2^2 - x_3^2)^{\frac{1}{2}}$$

Equation (4-1) are the dynamic equations for the nominal system. The actual system will contain additional unaccounted for perturbing forces which will cause the vehicle to depart from any predetermined nominal trajectory. A derivation of the control law for generating the nominal trajectory as well as the feedback guidance scheme based on the quasi-optimum control technique are given below.

4.2 Optimum Control Law

The Hamiltonian for the system (4-1) is given by

$$H = p_4(x_1a_0)^{3/2}/\mu^{\frac{1}{2}} + p'Fu - (p_5'kT(x_1a_0)^{3/2}/\mu^{\frac{1}{2}}) + p_6 \qquad (4\text{-}2)$$

where

$$p \equiv \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

It is easily verified that this Hamiltonian is maximized by the following control

$$u = \begin{cases} T\,\dfrac{F'p}{|F'p|} & \text{for } |F'p| \geq kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}} \\ \\ 0 & \text{for } |F'p| < kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}} \end{cases} \tag{4-3}$$

Substituting this optimum control into (4-2) results in the following Hamiltonian

$$H = H_{opt} = \begin{cases} (p_4(x_1a_0)^{3/2}/\mu^{\frac{1}{2}}) + T|F'p| - (kp_5T(a_0x_1)^{3/2}/\mu^{\frac{1}{2}}) + p_6 & \text{when } |F'p| \geq kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}} \\ \\ (p_4(x_1a_0)^{3/2}/\mu^{\frac{1}{2}}) + p_6 & \text{when } |F'p| < kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}} \end{cases} \tag{4-4}$$

The equations of motion together with the equations for the adjoint variables for the two cases (engine-on and engine-off) are obtained by direct application of (2-1) - (2-5).

<u>Case 1</u> Engine-off i.e., $|F'p| < kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}}$

$$\begin{aligned} &\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_5 = 0 \\ &\dot{x}_4 = (x_1a_0)^{3/2}/\mu^{\frac{1}{2}} \\ &\dot{x}_6 = 1 \\ &\dot{p}_1 = -\frac{3}{2}\,p_4(x_1a_0^3/\mu)^{\frac{1}{2}} \\ &\dot{p}_2 = \dot{p}_3 = \dot{p}_4 = \dot{p}_5 = \dot{p}_6 = 0 \end{aligned} \tag{4-5}$$

<u>Case 2</u> Engine-on i.e., $|F'p| \geq kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}}$

$$\dot{x} = TFF'p/|F'p|$$

$$\dot{x}_4 = (x_1a_0)^{3/2}/\mu^{\frac{1}{2}}$$

$$\dot{x}_5 = -kT(a_0x_1)^{3/2}/\mu^{\frac{1}{2}}$$

$$\dot{x}_6 = 1 \qquad (4\text{-}6)$$

$$^*\dot{p}_i = -(H_x)_i \qquad i = 1, 2, 3$$

$$\dot{p}_4 = 0 \qquad ; \qquad p_4 = \text{constant}$$

$$\dot{p}_5 = \frac{T}{x_5} |F'p|$$

$$^*\dot{p}_6 = -H_\tau$$

Since the only performance criterion being considered is that of minimum fuel or minimum time, in both Case 1 and Case 2 p_4 and the terminal value of p_5 will be specified as

$$p_4 = -1 \text{ and } p_5(t_f) = 0 \text{ for minimum time solution}$$

$$p_4 = \text{constant and } p_5(t_f) = +1 \text{ for minimum fuel solution}$$

For any set of boundary conditions of the form (2-2) together with the corresponding costate variables at the boundary expressed by (2-6) a nominal trajectory can be obtained by solving the resulting two-point boundary-value problem using any one of a number of iterative techiques.

Obtaining the optimum control law for the actual system will require, of course, knowing the equations governing the actual dynamic system. However, only the dynamics of the nominal system are known exactly and therefore we must make the assumption that the optimum control law for the actual system has the same functional relationship as for the nominal system; i.e., (4-3) is also the optimum control law for the actual system with the state and costate variables evaluated

---

* Explicit expressions for H_x and H_τ are given in section 4.3.

along the actual trajectory. The state variables along the actual path can be obtained from sensing devices while the costate can be approximated in accordance with (2-21). The correction matrix $M(t)$ is obtained by solving the matrix Riccati equation (2-17). In general this computation can either be done in real-time or else computed beforehand and stored with the nominal trajectory. Judging from the computation that would be required for calculating the matrices of second partials it appears that a pre-computed and stored correction matrix is the better choice. This will become more evident in the section 4.3.

4.3 Calculation of Coefficient Matrices H_{xp}, H_{px}, H_{xx}, H_{pp}:

The matrices of second partials required in the solution of the matrix Riccati equation (2-20) is obtained by direct differentiation of the Hamiltonian (4-4) for the two cases of "engine-on" and "engine-off".

Case 1 Engine-off i.e., $|F'p| < kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}}$

The first partials H_p and H_x are given below

$$H_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (x_1a_0)^{3/2}/\mu^{\frac{1}{2}} \\ 0 \\ 1 \end{bmatrix} \qquad H_x = \begin{bmatrix} 3/2\, p_4(x_1a_0^3/\mu^{\frac{1}{2}}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4-7}$$

The second partials are obtained by differentiation of (4-7) with respect to x and p. It is evident that all the elements of the H_{xp} matrix are zero except for $(H_{xp})_{41}$ which is given by

$$(H_{xp})_{41} = \frac{3}{2}(x_1a_0^3/\mu)^{\frac{1}{2}} \tag{4-8}$$

The (H_{xx}) matrix also has only one non-zero element

$$(H_{xx})_{11} = \frac{3}{4}\, p_1\left(\frac{a_0^3}{x_1\mu}\right)^{\frac{1}{2}} \tag{4-9}$$

All the elements of the H_{pp} matrix are zero.

Case 2 Engine-on i.e., $|F'p| \geq kp_5(x_1a_0)^{3/2}/\mu^{\frac{1}{2}}$

The F matrix defined in equation (4-1) contains x_1, x_2, x_3 and x_6 explicitly and therefore for the "engine-on" case it is convenient to first define the following set of 3 x 2 partial derivative matrices:

$$
\begin{aligned}
&Q^{(\ell)} \equiv \frac{\partial F}{\partial x_\ell} && \ell = 1, 2, 3, 6 \\
&Q^{(\ell j)} = Q^{(j\ell)} \equiv \frac{\partial^2 F}{\partial x_\ell \, \partial x_j} = \frac{\partial Q^{(\ell)}}{\partial x_j} && \ell = 1, 2, 3, 6 \\
& && j = 1, 2, 3, 6
\end{aligned}
\qquad (4\text{-}10)
$$

Expressions for these matrices are obtained by direct differentiation of the F matrix.

In terms of the $Q^{(\ell)}$ and $Q^{(\ell j)}$ matrices, we define another set $A^{(\ell)}$ and $A^{(\ell j)}$ which are given by

$$
A^{(\ell)} = [(Q^{(\ell)} F') + (Q^{(\ell)} F')'] \qquad \ell = 1, 2, 3, 6
$$

$$
A^{(\ell j)} = [(Q^{(\ell j)} F' + Q^{(\ell)} Q'^{(j)}) + (Q^{(\ell j)} F' + Q^{(\ell)} Q'^{(j)})'] \qquad (4\text{-}11)
$$

$$
\ell = 1, 2, 3, 6 \qquad j = 1, 2, 3, 6
$$

where $A^{(\ell)}$ and $A^{(\ell j)}$ are 3 x 3 matrices. The elements of H_p, H_x, H_{xp} and H_{pp} can be expressed in terms of these newly-defined matrices.

The six component H_p and H_x vectors are given by

$$
H_p = \begin{bmatrix} \hat{H}_p \\ \hdashline (x_1 a_0)^{3/2} \mu^{\frac{1}{2}} \\ - kT(a_0 x_1)^{3/2}/\mu^{\frac{1}{2}} \\ \hdashline 1 \end{bmatrix}
\begin{array}{l} \}\,3 \\ \\ \}\,2 \\ \}\,2 \end{array}
\qquad (4\text{-}12)
$$

where $\hat{H}_p \equiv \dfrac{TFF'p}{|F'p|}$

$$
H_x = \left[\begin{array}{c} \hat{H}_x \\ \hline 0 \\ 0 \\ \hline H_\tau \end{array}\right] + \begin{array}{c} \left[\begin{array}{c} \frac{3}{2}\,(x_1 a_0^3/\mu)^{\frac{1}{2}}\,(p_4 - k p_5 T) \\ 0 \\ 0 \\ \hline 0 \\ -\,T\,|F'p|/x_5 \\ \hline 0 \end{array}\right] \end{array} \begin{array}{l} \\ 3 \\ \\ \\ 2 \\ \\ 1 \end{array} \tag{4-13}
$$

where

$$
(\hat{H}_x)_i \equiv \frac{T}{2\,|F'p|}\; p'A^{(i)}\, p
$$

$$
H_\tau \equiv \frac{T}{2\,|F'p|}\; p'A^{(6)}\, p
$$

The second partials are obtained by differentiation of (4-12) and (4-13). This results in the following set of matrices

$$
H_{xp} = \begin{array}{c} \begin{array}{ccc} \quad 3 \quad & \quad 2 \quad & \quad 1 \quad \end{array} \\ \left[\begin{array}{ccc:cc:c} & \hat{H}_{xp} & & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & -\left[\dfrac{\hat{H}_p}{x_5}\right] & \left[H_{\tau p} \right] \\ \hdashline (h_{xp})_{41} & 0 & 0 & 0 & 0 & 0 \\ (h_{xp})_{51} & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ 3 \\ \\ 2 \\ \\ 1 \end{array} \tag{4-14}
$$

where

$$
(h_{xp})_{41} = \frac{3}{2}\,(x_1 a_0^3/\mu)^{\frac{1}{2}}
$$

$$
(h_{xp})_{51} = -\,\frac{3}{2}\,kT(x_1 a_0^3/\mu)^{\frac{1}{2}}
$$

The $\iota^{\underline{th}}$ column of the 3×3 $\hat{H}_{xp}$ matrix is given by

$$(H_{xp})_{\iota^{\underline{th}}\ \text{col.}} = \frac{T}{|F'p|} A^{(\iota)} p - \hat{H}_p p' A^{(\iota)} p/2|F'p|^2$$

The 3 - component $H_{\tau p}$ vector is given by

$$H_{\tau p} = \frac{T}{|F'p|} A^{(6)} p - \hat{H}_p p' A^{(6)} p/2|F'p|^2$$

$$H_{xx} = \begin{array}{c} \begin{array}{ccc} \xleftarrow{\quad 3 \quad} & \xleftarrow{\quad 2 \quad} & \xleftarrow{1} \end{array} \\ \left[\begin{array}{ccc:cc:c} & & & 0 & & \\ & H_{xx} & & 0 & \left[\dfrac{-H_x}{x_5}\right] & [H_{\tau x}] \\ & & & 0 & & \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ & [-\hat{H}'_x/x_5] & & 0 & (h_{xx})_{55} & -H_\tau/x_5 \\ \hdashline & [H'_{\tau x}] & & 0 & -H_\tau/x_5 & h_{\tau\tau} \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ \\ \\ 2 \\ \\ 1 \end{array} \qquad (4\text{-}15)$$

where

$$\hat{H}_{xx} \equiv \hat{R} + \hat{S} + \hat{T}$$

$$\hat{R} \equiv -\hat{H}_x \hat{H}'_x / T|F'p|$$

$$(\hat{S})_{\iota j} \equiv T_p' A^{(\iota j)} p/2|F'p| \qquad \begin{array}{l} \iota = 1, 2, 3 \\ j = 1, 2, 3 \end{array}$$

$$\hat{T} \equiv \frac{3}{4} \left(\frac{a_0^3}{x_1 \mu}\right)^{\frac{1}{2}} (p_1 - k p_5 T) \underbrace{\delta_{\iota 1} \delta_{j 1}}_{\substack{\text{zero except for} \\ (1,\ 1)^{th}\ \text{term}}}$$

The i^{th} component of the $H_{\tau x}$ vector is given by $(H_{\tau x})_i \equiv Tp'A^{(i6)} p/2|F'p| + H_\tau(\hat{H}_x)_i/T|F'p|$
$i = 1, 2, 3$

$$(h_{xx})_{55} = \frac{2T|F'p|}{x_5^2}$$

$$h_{\tau\tau} = \frac{2H_\tau^2}{T|F'p|} + \frac{T}{2|F'p|} p'A^{(66)} p$$

$$H_{pp} = \left[\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & \hat{H}_{pp} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{4-16}$$

$$\hat{H}_{pp} = \frac{T}{|F'p|}\left(FF' - \frac{(FF'p)(FF'p)'}{|F'p|^2}\right)$$

4.4 Initial Evaluation of the Correction Matrix $M(t_0)$

To be able to integrate the Ricatti equation (2-17) one must start with a value of the M matrix, at some determined time. From (2-9) we see that along the optimum nominal path the M matrix must satisfy the exact relationship

$$\dot{p} = M\dot{x}$$

It is clear that the above relationship only gives 6 equations (for a 6-component state vector) and yet $M(t_0)$ must be defined by 21 independent matrix elements (making use of the fact that M is a symmetric matrix). However, one can obtain a terminal value $M(T)$ by imbedding the boundary conditions into conditions on $M(T)$ and then integrating the Riccati equation backward to the initial time. The procedure for obtaining the matrix $M(T)$ is described in detail in reference [1]. Basically the technique is to express $M(t)$ in the form

$$M(t) = S(t) - R(t)\, Q^{-1}(t)\, R'(t) \tag{4-17}$$

where S, Q and R satisfy the following equation

$$
\begin{aligned}
-\dot{S} &= SH_{XP} + H_{PX}S + SH_{PP}S + H_{XX} \\
-\dot{R} &= (H_{PX} + SH_{PP})R \qquad\qquad (4\text{-}18) \\
-\dot{Q} &= R'H_{PP}R
\end{aligned}
$$

The following terminal conditions apply

$$S(T) = 0$$

$$
R(T) = \begin{array}{cc}
\left[\begin{array}{c|c} \Phi & -\dot{p}(T) \end{array}\right] & 6 \\
\begin{array}{cc} s & \quad 1 \end{array} &
\end{array}
\qquad\qquad (4\text{-}19)
$$

$$
Q(T) = \begin{array}{cc}
\left[\begin{array}{c|c} 0 & \Phi\dot{x}(T) \\ \hline \dot{x}'(T)\Phi' & -x'(T)\dot{p}(T) \end{array}\right] & \begin{array}{c} s \\ 1 \end{array} \\
\begin{array}{cc} s & \qquad 1 \end{array} &
\end{array}
$$

where

$$\Phi_{ij} = \frac{\partial \varphi_i}{\partial x_j} \qquad \begin{array}{l} i = 1, 2, \ldots, s \\ j = 1, 2, \ldots, n \end{array}$$

$$\varphi_i(x(T)) = 0 \qquad i = 1, 2, \ldots, s$$

are the boundary conditions

When s is less than or equal to one Q(T) is a non-singular matrix in which case M(T) can be evaluated directly by means of (4-17) and (4-19) without resorting to the differential equations (4-18).

Clearly when s is greater than one $Q(T)$ becomes a singular matrix. Consequently, $M(T)$ cannot be evaluated at $t = T$. This problem is circumvented by integrating the differential equations for S, R and Q for a small time Δ backwards from T and using the results to compute $M(T - \Delta)$.

For the specific case of a 6-component state vector with a single boundary condition (say on the terminal energy, or equivalently, on the semimajor axis of the trajectory) the $M(T)$ matrix can be expressed as

$$M(T) = \frac{1}{\dot{x}_1^2} \left[\begin{array}{c|ccccc} 2a_{11} - b & a_{21} & a_{31} & a_{41} & a_{51} & a_{61} \\ \hline a_{21} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & 0 & 0 & 0 & 0 & 0 \\ a_{61} & 0 & 0 & 0 & 0 & 0 \end{array} \right] \tag{4-20}$$

where

$$a_{ij} \equiv \dot{p}_i(T)\dot{x}_j(T)$$

$$b \equiv \dot{x}'(T)\dot{p}(T)$$

4.5 Discussion of Results

The quasi-optimum feedback quidance scheme was investigated for the case of a low thrust vehicle whose thrust vector is permantly misaligned by 10 degrees from its specified direction. A digital computer program was used to simulate the performance of the optimum guidance system, the open-loop guidance system and closed-loop or feedback system. The following initial conditions were used:

initial thrust acceleration = 0.05 ft/sec^2
exhaust velocity = 50,000 ft/sec
initial semimajor axis = 1.46 earth radii
initial eccentricity = .053

The duration of the simulated flight was 44.5 hrs. (approximately 9 $\frac{1}{2}$ orbits) at which time the semimajor axis reached 4.7 earth radii.

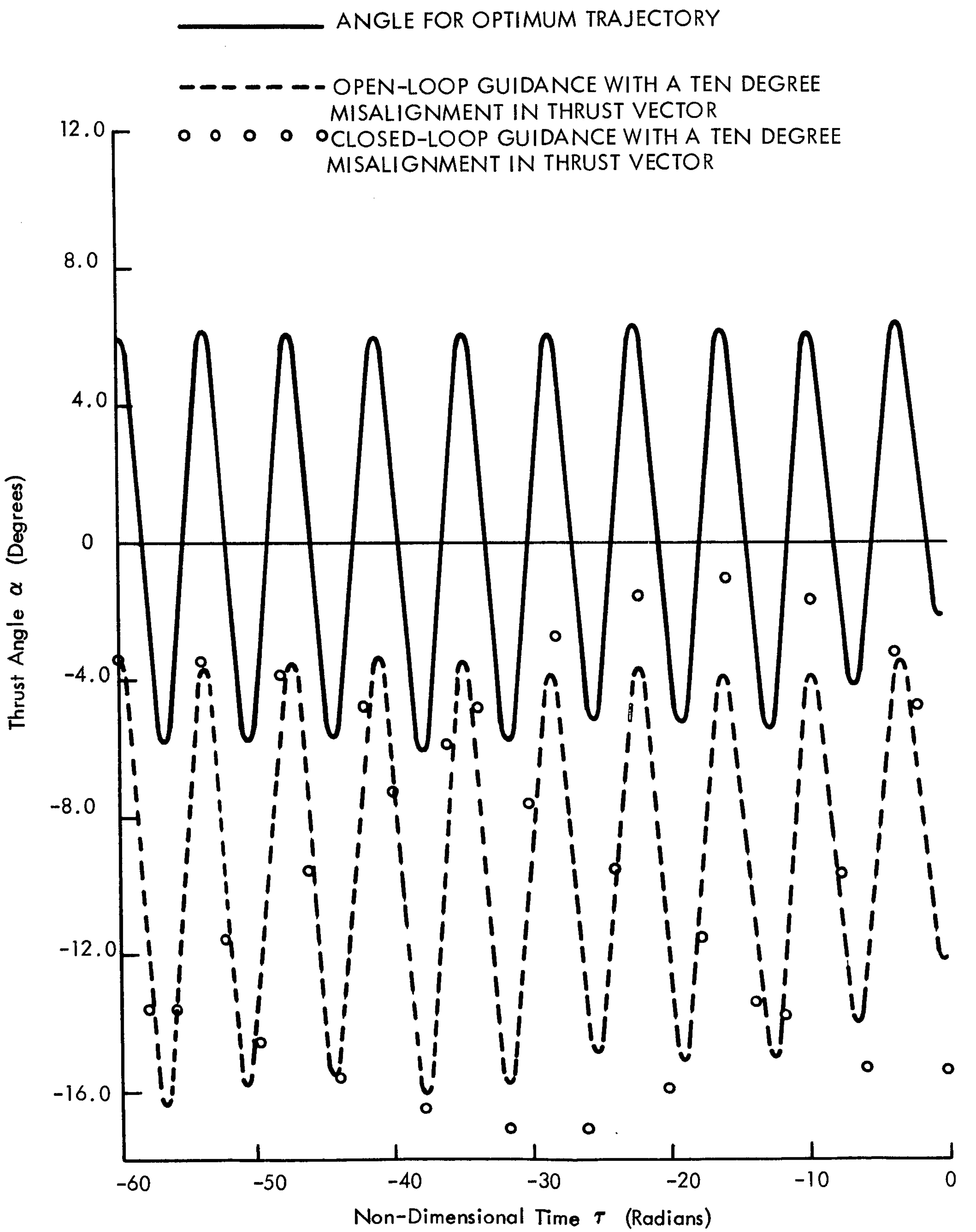
The first case that was considered was one in which only a single terminal condition was specified, namely the energy of the vehicle or equivalently the semimajor axis of the orbit. The optimum thrust vector was computed along the trajectory and then biased by 10 degrees to simulate the open-loop guidance system. In the closed-loop system the control was computed in accordance with (2-21) and (4-3) and then biased by 10 degrees to simulate the misalignment. The optimum direction of the thrust vector and those resulting from the open and closed-loop systems under a 10 degree misalignment is shown in Figure 4.1 . The open-loop thrust vector is, of course, exactly 10 degrees less than the optimum and the closed-loop thrust vector differs by less than 3 1/2 degrees from that of the open-loop system. A comparison of the performance of the three systems at the end of 44.5 hrs. is shown below in Table 4.1 . The variation of the semimajor axis during the entire flight is shown in Figure 4.2 .

Table 4.1

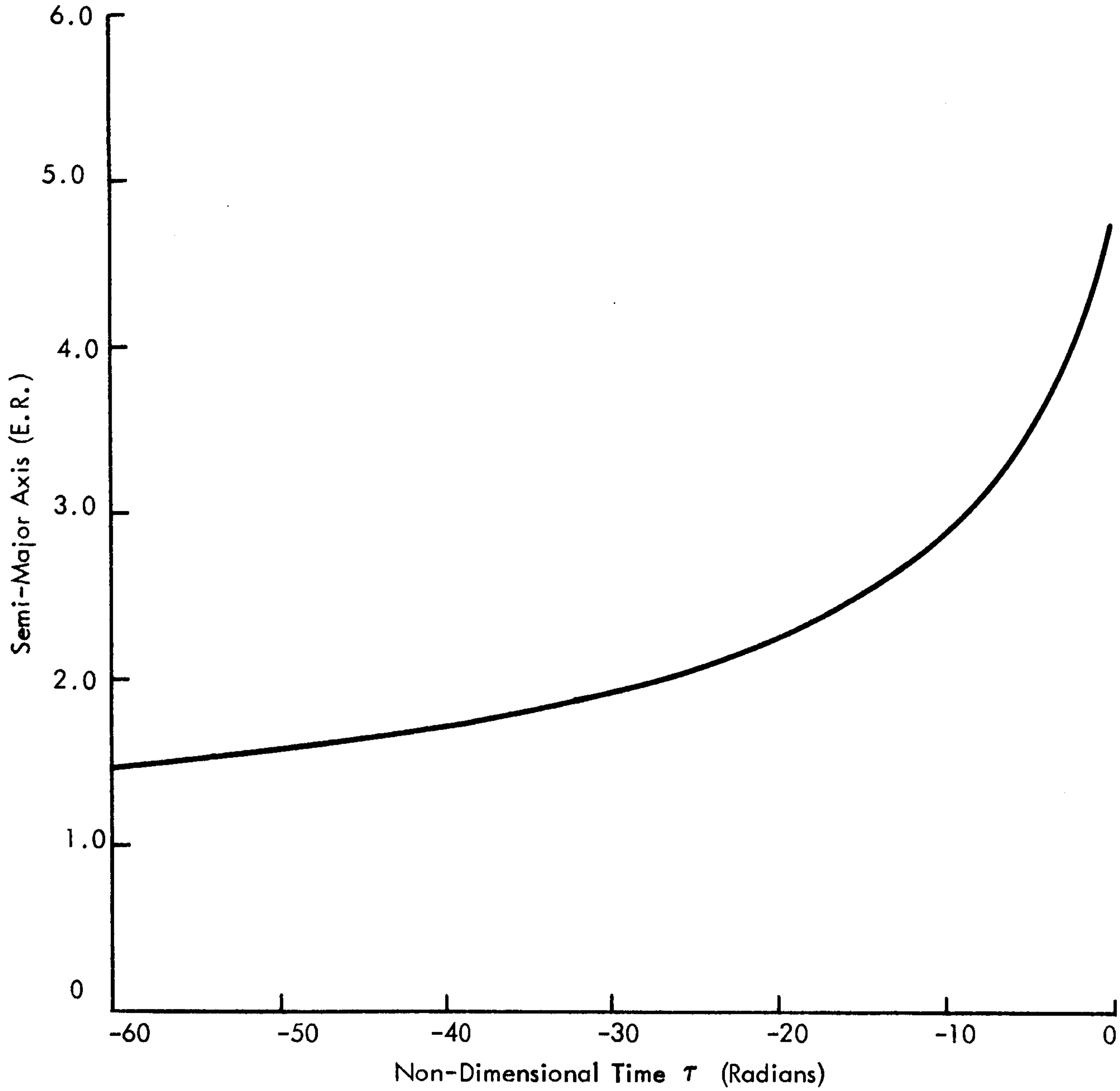
Summary of Performance After 44.5 Hours

case	semimajor axis (Earth Radii)
Optimum Thrust; No misalignment	4.5413
Feedback guidance; 10° misalignment	4.4387
Open-loop guidance; 10° misalignment	4.4373

These results indicate that for cases in which only the terminal energy of the vehicle (or equivalently semi major axis of the orbit) is specified the feedback guidance law provides a negligible improvement over the open-loop guidance system. However, this lack of improvement does not necessarily reflect a weakness in the feedback guidance law since the open-loop system is already fairly



DIRECTION OF THRUST VECTOR (WITH RESPECT TO LOCAL HORIZONTAL)
FIGURE 4-1



SEMI-MAJOR AXIS
FIGURE 4-2

close to optimum in achieving the desired terminal energy. In fact, a control consisting of pure circumferential thrust would probably be adequate if the energy of the vehicle was the only terminal condition to be satisfied. A more realistic situation, however, would require not only specifying a terminal energy, but also the orbital angle at which this energy is attained.

Specifying two terminal conditions (the semimajor axis and the orbital angle) requires the computation of a new initial feedback matrix $M(t_0)$ which in turn entails evaluation of $M(T)$ and a backward integration to t_0. However since $M(T)$ does not exist when more than one terminal condition is specified the value of M must be computed at time $T - \Delta$ using (4-17) - (4-19). For the two terminal conditions under consideration $x_1(T) = C_1$ and $x_6(T) = C_2$ the $R(T)$ and $Q(T)$ matrices specified in (4-19) are given by

$$R(T) = \begin{bmatrix} 1 & 0 & -\dot{p}_1(T) \\ 0 & 0 & -\dot{p}_2(T) \\ 0 & 0 & -\dot{p}_3(T) \\ 0 & 0 & -\dot{p}_4(T) \\ 0 & 0 & -\dot{p}_5(T) \\ 0 & 1 & -\dot{p}_6(T) \end{bmatrix}, \qquad Q(T) = \begin{bmatrix} 0 & 0 & \dot{x}_1(T) \\ 0 & 0 & \dot{x}_6(T) \\ \dot{x}_1(T) & \dot{x}_6(T) & -\sum_{i=1}^{6} \dot{x}_i(T)\dot{p}_i(T) \end{bmatrix}$$

In one computer run the S, Q and R equations were integrated from T to $T - 0.5$, at which point M was evalutated and the remainder of the trajectory computed using the costate equation $\dot{p} = M\dot{x}$. It turns out that the computation of M this close to the terminal time produces elements of the matrix which are very large (of the order of 10^{10}). The values of $\dot{x}$ and $\dot{p}$ are of the order of magnitude of unity which implies that even 7 place accuracy in this computation of M can result in errors of as much as 3 orders of magnitude in the costate vector p.

In another run the equations were integrated backwards to $T - 45.0$ and the M matrix computed of this point. The elements of the matrix were of the order of 10^5 indicating that 7 place accuracy in this computation of M might suffice. However with the algorithm presently being used to compute $M(t_0)$ the normal digital computer round-off and truncation errors would be intolerable. Therefore a reprogramming of the algorithm to minimize the computational errors is required before any conclusive results can be obtained.

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APPENDIX I

IGM Guidance Law

The equations used to simulate the IGM guidance law were obtained from References 6, 7, and 8. They are divided into 3 sections which are:

(1) Burnout State Predictor

(2) Time-to-go Computation

(3) Steering Angle Computation

] The equations used are illustrated in Figure I-1. The data required to initialize the algorithm is listed below:

F = thrust in pounds

T_{GO} = time-to-go

$\dot{m}$ = mass flow rate

Y_{DES} = burnout altitude

$\dot{X}_{DES}$ = x - velocity at burnout

$\dot{Y}_{DES}$ = y - velocity at burnout

g_{DES} = gravitation acceleration at burnout altitude

Δt = time between guidance law computations

WFTR = weighting factor for improving burnout angle prediction when long and/or steep trajectories are encountered.

In addition the following inertial measurements are required.

1) vehicle mass
2) x and y position
3) x and y velocity
4) $\Phi = \tan^{-1}(y/x)$
5) $R = \sqrt{x^2 + y^2}$

Figure I - 1 IGM Guidance Equations

INITIAL DATA

μ

$F,\ T_{GO},\ \dot{m},\ Y_{DES},\ \dot{X}_{DES},\ \dot{Y}_{DES},\ g_{DES},\ \Delta t,\ \text{WFTR}$

↓

INERTIAL DATA

$m,\ x,\ y,\ \dot{x},\ \dot{y},\ R,\ \Phi$

↓

BURNOUT STATE PREDICTOR

$$V_{ex} = F/m$$

$$\tau_0 = m/\dot{m}$$

$$T_{GO} = T_{GO} - \Delta t$$

$$A_1 = -V_{ex} \ln_e \left(\frac{\tau_0 - T_{GO}}{\tau_0} \right)$$

$$A_2 = A_1 \tau_0 - V_{ex} T_{GO}$$

$$A_3 = -A_2 + A_1 T_{GO}$$

$$A_4 = \tau_0 A_3 - V_{ex} T_{GO}^2/2$$

$$\Phi_{T_{GO}} = (A_3 + T_{GO}\sqrt{\dot{x}^2 + \dot{y}^2} + T_{GO}^2\,\text{WFTR})/Y_{DES}$$

$$\Phi_{BO} = \Phi + \Phi_{T_{GO}}$$

$$x_{BO} = x \cos \Phi_{BO} - y \sin \Phi_{BO}$$

$$y_{BO} = x \sin \Phi_{BO} + y \cos \Phi_{BO}$$

$$\dot{x}_{BO} = \dot{x} \cos \Phi_{BO} - \dot{y} \sin \Phi_{BO}$$

$$\dot{y}_{BO} = \dot{x} \sin \Phi_{BO} + \dot{y} \cos \Phi_{BO}$$

$$g = \mu/R^2$$

$$g^* = (g + g_{DES})/2$$

$$\Phi^* = \Phi_{T_{GO}}/2$$

↓

(A)

A

TIME-TO-GO COMPUTATION

$$V_{g_x}^2 = (\dot{X}_{DES} - \dot{x}_{BO} - g^* T_{GO} \sin \Phi^*)^2$$

$$V_{g_y}^2 = (\dot{Y}_{DES} - \dot{y}_{BO} + g^* T_{GO} \cos \Phi^*)^2$$

$$G = \frac{1}{2} \left(\frac{V_{g_x}^2 + V_{g_y}^2}{V_{ex} \ln_e \left[\frac{\tau_0}{\tau_0 - T_{GO}} \right]} - V_{ex} \ln_e \left[\frac{\tau_0}{\tau_0 - T_{GO}} \right] \right)$$

$$\Delta T = G(\tau_0 - T_{GO})/V_{ex}$$

$$T_{GO} = T_{GO} + \Delta T$$

STEERING ANGLE COMPUTATION

$$A_1 = - V_{ex} \ln_e \left(\frac{\tau_0 - T_{GO}}{\tau_0} \right)$$

$$A_2 = A_1 \tau_0 - V_{ex} T_{GO}$$

$$A_3 = - A_2 + A_1 T_{GO}$$

$$A_4 = \tau_0 A_3 - V_{ex} T_{GO}^2 / 2$$

$$\tilde{\chi} = \tan^{-1} \left[\frac{\dot{Y}_{DES} - \dot{y}_{BO} + g^* T_{GO} \cos \Phi^*}{\dot{X}_{DES} - \dot{x}_{BO} - g^* T_{GO} \sin \Phi^*} \right]$$

$$P = A_3 \cos \tilde{\chi}$$

$$Q = A_4 \cos \tilde{\chi}$$

$$R = Y_{DES} - y_{BO} - \dot{y}_{BO} T_{GO} + \frac{1}{2} g^* T_{GO}^2 \cos \Phi^* - A_3 \sin \tilde{\chi}$$

$$a = \frac{A_2 R}{A_1 Q - A_2 P}$$

$$b = a A_1 / A_2$$

$$\chi = \tilde{\chi} - a + bt$$

APPENDIX II

Modified Secant Algorithm for N Equations with N Unknowns

In order to solve approximately the n equations

$$E_1(x_1, x_2, \dots, x_n) = 0$$
$$E_2(x_1, x_2, \dots, x_n) = 0$$
$$\vdots$$
$$E_n(x_1, x_2, \dots, x_n) = 0$$

Assume that for $n+1$ points the values of E_l are known. Associated with each of the $n+1$ points are $n+1$ sets of equations

$$\left(E_1^{(1)}, E_2^{(1)}, \dots, E_n^{(1)}\right) \dots \left(E_1^{(n+1)}, E_2^{(n+1)}, \dots, E_n^{(n+1)}\right)$$

For each set of equations form the sum

$$\Sigma_1 = \sum_{l=1}^{n} (E_l^{(1)})^2, \dots \Sigma_{n+1} = \sum_{l=1}^{n} (E_l^{(n+1)})^2$$

Denote the largest sum Σ_{max} and the smallest sum Σ_{min}. Set K equal to a number between zero and one. The following set of equations is solved for q_l.

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ E_1^{(1)} & E_1^{(2)} & \dots & E_1^{(n+1)} \\ E_2^{(1)} & E_2^{(2)} & \dots & E_2^{(n+1)} \\ \vdots & \vdots & & \vdots \\ E_{n+1}^{(1)} & E_{n+1}^{(2)} & \dots & E_{n+1}^{(n+1)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ (1-K)\,E_1^{(min)} \\ (1-K)\,E_2^{(min)} \\ \vdots \\ (1-K)\,E_n^{(min)} \end{bmatrix} \qquad (II-1)$$

With these q_i's compute a new point as follows

$$\begin{bmatrix} x_1^{(n+2)} \\ x_2^{(n+2)} \\ \cdot \\ \cdot \\ \cdot \\ x_n^{(n+2)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n+1)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(n+1)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n+1)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_{n+1} \end{bmatrix} \qquad \text{(II - 2)}$$

Using the new point, x^{n+2}, calculate $E_i^{(n+2)}$ and $\Sigma_{new} = \sum_{i=1}^{n} (E_i^{(n+2)})^2$. If $\Sigma_{new} > \Sigma_{max}$ halve k and repeat (II - 1 and 2). If $\Sigma_{new} \le \Sigma_{max}$ double k (up to a maximum of 1), replace Σ_{max} with Σ_{new} in (II-1), replace $x^{(max)}$ with $x^{(n+2)}$ in (II-2) and repeat the computation of equations (II-1 and 2). The algorithm is terminated when Σ_{min} is less than a predetermined value and the solution is taken as $x^{(min)}$.

APPENDIX III

Calculation of Gravity Polynomial Coefficients for the PGE Guidance Law

The algorithm used to calculate the coefficients for the gravity polynomials was supplied by MSFC and is illustrated in Figure III-1 .

Figure III - 1 Equations Used to Calculate the Coefficients for the PGE Gravity Polynomials

INERTIAL DATA

x_ℓ , $\dot{x}_\ell$, m

$(\ell = 1, 2, 3)$

CURRENT ESTIMATED ADJOINT VARIABLES AND TIME-TO-GO

P_ℓ , $\dot{P}_\ell$, T_{GO} ,

$(\ell = 1, 2, 3)$

↓

$$C = \sum_{\ell=1}^{3} P_\ell^2 \qquad \ell = 1, 2, 3$$

$$A = -\sum_{\ell=1}^{3} \frac{P_\ell \dot{P}_\ell}{C}$$

$$B^2 = \sum_{\ell=1}^{3} \frac{\dot{P}_\ell^2}{C}$$

$$Z = 1 + 2AT_{GO} + B^2 T_{GO}^2$$

$$\gamma = \dot{m}/m$$

$$R = \left[\sum_{\ell=1}^{3} x_\ell^2\right]^{1/2}$$

$$V = \left[\sum_{\ell=1}^{3} \dot{x}_\ell^2\right]^{1/2}$$

↓

(A)

A

$$\dot{R} = \left[\sum_{i=1}^{3} \dot{x}_i x_i\right] \Big/ R$$

$$\ddot{x}_i = \frac{F}{m\sqrt{Z}} P_i - \mu \frac{x_i}{R^3}$$

$$\dddot{x}_i = \frac{F}{m\sqrt{Z}} (\dot{P}_i - P_i\{\gamma + A\}) - \frac{\mu}{R^3} (\dot{x}_i - 3x_i \dot{R}/R)$$

$$\ddot{R} = \{V^2 - \dot{R}^2 + \sum_{i=1}^{3} x_i \ddot{x}_i\}/R$$

$$\dddot{R} = \{3(\sum_{i=1}^{3} \dot{x}_i \ddot{x}_i - \dot{R}\ddot{R}) + \sum_{i=1}^{3} x_i \dddot{x}_i\}/R$$

$$C_{i1} = x_i/R^3$$

$$C_{i2} = (\dot{x}_i - 3R^2\dot{R} C_{i1})/R^3$$

$$C_{i3} = \left(\frac{\ddot{x}_i}{2} - 3R^2\dot{R}C_{i2} - 3R\{R\frac{\ddot{R}}{2} + \dot{R}^2\} C_{i1}\right) \Big/ R^3$$

$$C_{i4} = \left(\frac{\dddot{x}_i}{6} - 3R^2\dot{R}C_{i3} - 3R\{R\frac{\ddot{R}}{2} + \dot{R}^2\} C_{i2} - \{3R\dot{R}\ddot{R} + \dot{R}^3 + \frac{\dddot{R}}{3}R^2\} C_{i1}\right) \Big/ R^3$$

$$g_i = \mu \sum_{j=1}^{4} C_{ij} T_{GO}^{j-1} \qquad (i = 1, 2, 3)$$